ON THE IWASAWA INVARIANTS OF ARTIN REPRESENTATIONS

ADITYA KARNATAKI AND ANWESH RAY

ABSTRACT. We study Iwasawa invariants associated to Selmer groups of Artin representations, and criteria for the vanishing of the associated algebraic Iwasawa invariants. The conditions obtained can be used to study natural distribution questions in this context.

1. INTRODUCTION

Let p be an odd prime number and \mathbb{Z}_p denote the ring of p-adic integers. Let K be a number field and fix an algebraic closure \overline{K} of K. A \mathbb{Z}_p -extension of K is an infinite Galois extension K_{∞}/K such that the Galois group $\operatorname{Gal}(K_{\infty}/K)$ is isomorphic to \mathbb{Z}_p as a topological group. Let $K_n \subset K_\infty$ be the extension of K for which $\operatorname{Gal}(K_n/K) \simeq \mathbb{Z}/p^n\mathbb{Z}$, and $h_k(K_n)$ denote the cardinality of the p-primary part of the class number of K_n . Writing $h_p(K_n) = p^{e_n}$, Iwasawa [Iwa59] showed that for all large enough values of n,

$$e_n = p^n \mu + n\lambda + \nu,$$

where $\mu, \lambda \in \mathbb{Z}_{\geq 0}$ and $\nu \in \mathbb{Z}$ are invariants that depend on the extension K_{∞}/K and not on *n*. The results of Iwasawa motivated Mazur [Maz72] to study the growth properties of the *p*-primary Selmer groups of *p*-ordinary abelian varieties in \mathbb{Z}_p -extensions. Mazur's results were later extended to very general classes of ordinary Galois representations by Greenberg, cf. [G⁺89].

Another class of representations that are natural to consider are Artin representations. Let K/\mathbb{Q} be a finite and totally imaginary Galois extension with Galois group $\Delta := \operatorname{Gal}(K/\mathbb{Q})$, and let $\rho : \Delta \to \operatorname{GL}_d(\overline{\mathbb{Q}})$ be an irreducible Artin representation. Let p be an odd prime number and let $\sigma_p : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ be an embedding and via σ_p we view ρ as a representation $\rho : \Delta \to \operatorname{GL}_d(\overline{\mathbb{Q}}_p)$. Let v denote an archimedian prime of K, and set K_v to denote the v-adic completion of K. We shall identify $\operatorname{Gal}(K_v/\mathbb{R})$ with a subgroup Δ_v of Δ . Set d^+ to denote the multiplicity of the trivial character in $\rho_{|\Delta_v|}$ and observe that this number is well defined and independent of the choice of the archimedian prime v. When K is totally real, we find that $d^+ = d$. Let \mathfrak{p} be a prime of K that lies above p, and let $\Delta_{\mathfrak{p}} \subset \Delta$ be the decomposition group \mathfrak{p} . Following Greenberg and Vatsal [GV20], we make the following assumptions.

Assumption 1.1. With respect to notation above, assume that

(1) p does not divide $[K : \mathbb{Q}]$,

(2) $d^+ = 1$,

²⁰²⁰ Mathematics Subject Classification. 11R23, 11R34, 11F80 (Primary).

Key words and phrases. Iwasawa theory, Artin representations, Euler characteristic, distribution questions.

A. KARNATAKI AND A. RAY

(3) there exists a 1-dimensional representation $\epsilon_{\mathfrak{p}}$ of $\Delta_{\mathfrak{p}}$ that occurs with multiplicity 1 in $\rho_{|\Delta_{\mathfrak{p}}}$.

In the special case d = 2 and $d^+ = 1$, such representations are expected to arise from Hecke eigenforms of weight 1 on $\Gamma_1(N)$, where N is the Artin conductor of ρ . The conjecture has been settled in various special cases, cf. [BDSBT01, Tay03] and references therein. The choice of the character $\epsilon_{\mathfrak{p}}$ plays a role in the definition of the Selmer groups associated to ρ . Take \mathcal{K} be the completion of K at \mathfrak{p} . There is a natural isomorphism $\operatorname{Gal}(\mathcal{K}/\mathbb{Q}_p) \simeq \Delta_{\mathfrak{p}}$, and set ϵ to denote the composite

$$\operatorname{Gal}(\mathcal{K}/\mathbb{Q}_p) \xrightarrow{\sim} \Delta_{\mathfrak{p}} \xrightarrow{\epsilon_{\mathfrak{p}}} \overline{\mathbb{Q}}^{\times}.$$

Let χ denote the character associated to ρ and $\mathbb{Q}(\chi)$ denote the field generated by the values of χ . Let \mathcal{F} be the field generated by \mathbb{Q}_p , the values of χ and the values of ϵ_p . We regard ρ as a representation a vector space V defined over \mathcal{F} and note that $\dim_{\mathcal{F}} V = d$. Let V^{ϵ_p} be the the maximal \mathcal{F} -subspace of V on which Δ_p acts by ϵ_p . By hypothesis, $\dim_{\mathcal{F}} V^{\epsilon_p} = 1$. Let \mathcal{O} be the valuation ring in \mathcal{F} and ϖ be its uniformizer. We assume that ρ is irreducible, and therefore, there is a Galois stable \mathcal{O} -lattice $T \subset V$. This lattice is uniquely determined up to scaling by a constant. We consider the divisible Galois module D := V/T and let D^{ϵ_p} be the image of V^{ϵ_p} in D. Let \mathbb{Q}_{∞} denote the cyclotomic \mathbb{Z}_p -extension of \mathbb{Q} . Let $\mathbb{Q}_n \subset \mathbb{Q}_\infty$ be the subextension such that $\operatorname{Gal}(\mathbb{Q}_n/\mathbb{Q}) \simeq \mathbb{Z}/p^n\mathbb{Z}$.

$$S_{\chi,\epsilon}(\mathbb{Q}_n) := \ker \left\{ H^1(\mathbb{Q}_n, D) \longrightarrow \prod_{\nu \nmid p} H^1(\mathbb{Q}_{n,\nu}^{\mathrm{nr}}, D) \times H^1(\mathbb{Q}_{n,\eta_p}^{\mathrm{nr}}, D/D^{\epsilon_{\mathfrak{p}}}) \right\},\$$

where η_p is the unique prime that lies above p.

Theorem 1.2 (Greenberg and Vatsal). Suppose that the Assumption 1.1 holds. Then, the Selmer group $S_{\chi,\epsilon}(\mathbb{Q}_n)$ is finite for all $n \in \mathbb{Z}_{\geq 0}$.

Proof. The above result is [GV20, Proposition 3.1].

Definition 1.3. Define the Selmer group over \mathbb{Q}_{∞} to denote the direct limit with respect to restriction maps

$$S_{\chi,\epsilon}(\mathbb{Q}_{\infty}) := \varinjlim_n S_{\chi,\epsilon}(\mathbb{Q}_n).$$

Greenberg and Vatsal show that the Selmer group is cofinitely generated and cotorsion over the Iwasawa algebra, which is a formal power series ring over \mathcal{O} in 1-variable. Leveraging the results of Greenberg and Vatsal, we study the Euler characteristic formula associated to these Selmer groups and utilize it to study explicit conditions for the vanishing of Selmer group $S_{\chi,\epsilon}(\mathbb{Q}_{\infty})$. For instance, we are able to prove that there is an explicit relationship between the vanishing of the Selmer group and the *p*-rationality of K (in the sense of [MD90]).

Theorem 1.4 (Theorem 3.11). Assume that

- (i) the conditions of Assumption 1.1 are satisfied.
- (*ii*) $H^0(\Delta_{\mathfrak{p}}, D/D^{\epsilon_{\mathfrak{p}}}) = 0$,
- (iii) K is p-rational,
- (iv) p does not divide the class number of K.

Then, $S_{\chi,\epsilon}(\mathbb{Q}_{\infty}) = 0.$

These criteria are illustrated in two special cases, namely that of 2-dimensional irreducible Artin representations (cf. Theorem 3.12) and 3-dimensional icosahedral Artin representations of A_5 (cf. Theorem 3.13). The Theorem 3.9 gives a more refined criterion that is equivalent to the vanishing of $S_{\chi,\epsilon}(\mathbb{Q}_{\infty})$. The relationship between *p*-rationality and the Iwasawa invariants of number fields has been studied by Hajir and Maire, cf. [HM19].

For simplicity, we then specialize our discussion to 2-dimensional Artin representation of dihedral type. Let L is an imaginary quadratic field and $\zeta : G_L \to \overline{\mathbb{Q}}^{\times}$ is a character and $\rho = \operatorname{Ind}_{L}^{\mathbb{Q}} \zeta : G_{\mathbb{Q}} \to \operatorname{GL}_{2}(\overline{\mathbb{Q}})$ be the associated Artin representation. Set ζ' to be conjugate character defined by setting $\zeta'(x) = \zeta(cxc^{-1})$, where c denotes complex conjugation. Assume that $\zeta' = \zeta^{-1}$, thus ρ is dihedral type. Let $S(\rho)$ be the set of primes p such that

- (1) p is odd and $p \nmid [K : \mathbb{Q}],$
- (2) p splits in L, $p\mathcal{O}_L = \pi\pi^*$, and π is inert in K/L.

It is easy to see that if $\mathfrak{p}|p$ is a prime of K, then, the conditions of Assumption of 1.1 are satisfied. Let $T(\rho)$ be the set of primes $p \in S(\rho)$ such that the Selmer group $S_{\chi,\epsilon}(\mathbb{Q}_{\infty})$ vanishes for all of the primes \mathfrak{p} that lie above p. Set $T'(\rho) := S(\rho) \setminus T(\rho)$. In section 4, we apply our results to study the following natural question.

Question 1.5. What can be said about the densities of the sets of primes in $T(\rho)$ and $T'(\rho)$?

A conjecture of Gras predicts that any number field K is p-rational at all but finitely many primes p. This has conjectural implications to the vanishing of Selmer groups for all but finitely many primes a compatible family of Artin representations. It the case when K/\mathbb{Q} is an S_3 -extension, we prove certain unconditional results by leveraging a result of Maire and Rougnant [MR20] on the p-rationality of S_3 -extensions of \mathbb{Q} .

Theorem 1.6 (Theorem 4.10). Let K/\mathbb{Q} be an imaginary S_3 extension and ρ be a 2dimensional Artin representation that factors through $\operatorname{Gal}(K/\mathbb{Q})$. Then,

$$#\{p \le x \mid p \in T(\rho)\} \ge c \log x.$$

The result above is certainly weaker than what one is led to conjecture, however, it does well to illustrate the effectiveness of the Theorem 3.11. Throughout, we motivate our results by drawing upon analogues from the classical Iwasawa theory of class groups.

1.1. **Organization.** Including the introduction, the article consists of four sections. In section 2 we review the classical Iwasawa theory of class groups and Artin representations. We review results of Federer and Gross [FG80] which gives an explicit relationship between p-adic regulators and Iwasawa invariants. We end this section by reviewing some results of Greenberg and Vatsal [GV20] on the Iwasawa theory of Artin representations. In the section 3, we discuss the notion of the Euler characteristic of a cofinitely generated and cotorsion module over the Iwasawa algebra. In section 4, we formulate and study some natural distribution questions, which serve to illustrate our results.

A. KARNATAKI AND A. RAY

2. Iwasawa theory of class groups and Artin representations

2.1. Classical Iwasawa theory and Iwasawa invariants. We review the classical Iwasawa theory over the cyclotomic \mathbb{Z}_p -extension of a number fields. The reader may refer to [Was97] for a more comprehensive treatment. Throughout this section, we shall set K to denote a number field and p an odd prime number. Fix an algebraic closure \overline{K} of K. All algebraic extensions considered in this article shall implicitly be assumed to be in \overline{K} . Let $\operatorname{Cl}(K)$ denote the class group of K and $\operatorname{Cl}_p(K)$ its p-primary part.

For $n \geq 1$, let $K(\mu_{p^n})$ be the number field extension of K obtained by adjoining the p^n -th roots of unity to K. Denote by $K(\mu_{p^\infty})$ the union of all extensions $K(\mu_{p^n})$. There is a unique \mathbb{Z}_p -extension K_{∞}/K which is contained in $K(\mu_{p^\infty})$, which is referred to as the cyclotomic \mathbb{Z}_p -extension of K. For $n \geq 1$, set K_n/K to be the sub-extension of K_{∞}/K , for which $\operatorname{Gal}(K_n/K)$ is isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$. Setting $K_0 := K$, refer to K_n as the *n*-th layer. Taking $\Gamma_n := \operatorname{Gal}(K_{\infty}/K_n)$ we identify $\operatorname{Gal}(K_n/K)$ with Γ/Γ_n . For $n \in \mathbb{Z}_{\geq 0}$, denote by $H_p(K_n)$ the p-Hilbert class field of K_n . In other words, $H_p(K_n)$ is the maximal unramified abelian p-extension of K_n in \overline{K} . Set X_n to denote the Galois group $\operatorname{Gal}(H_p(K_n)/K_n)$, and identify X_n with the maximal p-primary quotient of $\operatorname{Cl}(K_n)$. Thus, $[H_p(K_n) : K_n]$ is equal to $\#\operatorname{Cl}_p(K_n)$. Since the primes above p are totally ramified in K_{∞} , it follows that $H_p(K_n) \cap K_{\infty} = K_n$, and thus, there are surjective maps $X_m \to X_n$ for all $m \geq n$. Taking X_{∞} to be the inverse limit $\varprojlim_n X_n$, we find that X_{∞} is both a \mathbb{Z}_p -module as well as a module over Γ . On the other hand, letting $H_p(K_{\infty})$ denote the maximal unramified abelian pro-p extension of K_{∞} , we identify X_{∞} with the Galois group $\operatorname{Gal}(H_p(K_{\infty})/K_{\infty})$.

On order to better study the algebraic structure of X_{∞} , it proves fruitful to view X_{∞} as a module over a completed group ring known as the Iwasawa algebra. This algebra is defined as follows

$$\Lambda := \mathbb{Z}_p[\![\Gamma]\!] = \varprojlim_n \mathbb{Z}_p[\Gamma/\Gamma_n].$$

Choosing a topological generator $\gamma \in \Gamma$, we identify Λ with the formal power series ring $\mathbb{Z}_p[\![T]\!]$, by setting $T := (\gamma - 1)$. As a Λ -module, X_{∞} is finitely generated and torsion [Was97, chapter 13].

Let \mathcal{O} be a valuation ring with residue characteristic p, and let ϖ be a uniformizer of \mathcal{O} . Then, the Iwasawa algebra over \mathcal{O} is defined by extending coefficients to \mathcal{O} , as follows $\Lambda_{\mathcal{O}} := \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}$. The Iwasawa algebra $\Lambda_{\mathcal{O}}$ is a local ring with maximal ideal $\mathfrak{m} = (\varpi, T)$. A polynomial $f(T) \in \mathcal{O}[T]$ is said to be *distinguished* if it is a monic polynomial whose non-leading coefficients are divisible by ϖ . The Weierstrass preparation theorem states that any power series f(T) decomposes into a product

$$f(T) = \varpi^{\mu} \times g(T) \times u(T),$$

where $\mu \in \mathbb{Z}_{\geq 0}$, g(T) is a distinguised polynomial and u(T) is a unit in $\Lambda_{\mathcal{O}}$. The μ invariant of f(T) is the power of ϖ above, and the λ -invariant is the degree of the distinguished polynomial g(T). The prime ideals of height 1 are the principal ideals (ϖ) and (g(T)), where g(T) is an irreducible distinguished polynomial. Given a finitely generated and torsion $\Lambda_{\mathcal{O}}$ -module M, there is a homomorphism of $\Lambda_{\mathcal{O}}$ -modules

(2.1)
$$M \longrightarrow \left(\bigoplus_{i=1}^{s} \frac{\mathcal{O}\llbracket T \rrbracket}{(\varpi^{\mu_i})} \right) \oplus \left(\bigoplus_{j=1}^{t} \frac{\mathcal{O}\llbracket T \rrbracket}{(f_i(T)^{\lambda_i})} \right),$$

with finite kernel and cokernel. Here, $\mu_i, \lambda_j \ge 0$, and $f_j(T)$ are irreducible distinguished polynomials. For further details, we refer to [Was97, Theorem 3.12].

Definition 2.1. The μ -invariant $\mu_p(M)$ is the sum of the entries $\sum_{i=1}^{s} \mu_i$ if s > 0, and is set to be 0 if s = 0. On the other hand, the λ -invariant $\lambda_p(M)$ is defined to be $\sum_{i=1}^{s} \lambda_i \deg(f_i)$ if s > 0, and defined to be 0 if t = 0. The characteristic element $f_M(T)$ is the product

$$f_M(T) := \prod_i \varpi^{\mu_i} \times \prod_j f_j(T)^{\lambda_j}$$

We remark that the μ -invariant and λ -invariant of $f_M(T)$ are $\mu_p(M)$ and $\lambda_p(M)$ respectively.

Proposition 2.2. Suppose that M is a finitely generated and torsion $\Lambda_{\mathcal{O}}$ -module. Then, the following assertions hold,

- (1) $\mu_p(M) = 0$ if and only if M is finitely generated as an O-module. In this case, $\lambda_p(M)$ is the O-rank of M.
- (2) Letting $r_p(M)$ denote the order of vanishing of f_M at 0, we have that

$$\lambda_p(M) \ge r_p(M).$$

(3) Write $f_M(T)$ as a power series

$$f_M(T) = a_r T^r + a_{r+1} T^{r+1} + \dots + a_\lambda T^\lambda,$$

where $r = r_p(M)$ and $\lambda = \lambda_p(M)$. The μ -invariant $\mu_p(M) = 0$ if and only if there is a coefficient a_i not divisible by ϖ .

(4) We have that $\varpi^{\mu} || a_{\lambda}$ and that $\varpi^{\mu+1} || a_i$ for all $i < \lambda$.

Proof. The results are easy observations that follow from the structural homomorphism (2.1) and the definition of the Iwasawa invariants.

Remark 2.3. Let \mathcal{O}' be a valuation ring that is a finite extension of \mathcal{O} , and e be its ramification index. Setting $M_{\mathcal{O}'} := M \otimes_{\mathcal{O}} \mathcal{O}'$ and regard $M_{\mathcal{O}'}$ as a module over $\Lambda_{\mathcal{O}'}$. Then, it is easy to see that

$$\mu_p(M_{\mathcal{O}'}) = e\mu_p(M) \text{ and } \lambda_p(M_{\mathcal{O}'}) = \lambda_p(M).$$

We denote the μ -invariant (resp. λ -invariant) of X_{∞} by $\mu_p(K)$ (resp. $\lambda_p(K)$). In this setting, $\mathcal{O} := \mathbb{Z}_p$. Iwasawa proved that for all large enough values of n, there is an invariant $\nu_p(K) \in \mathbb{Z}$ for which

$$\log_p\left(\#\operatorname{Cl}_p(K_n)\right) = p^n \mu_p(K) + n\lambda_p(K) + \nu_p(K),$$

cf. [Was97, Theorem 13.13]. Moreover, Iwasawa conjectured that $\mu_p(K) = 0$ for all number fields K, cf. [Iwa73]. For abelian extensions K/\mathbb{Q} the conjecture has been proven by Ferrero and Washington [FW79].

2.2. The leading coefficient of the characteristic series. Let K be a CM field with totally real subfield K^+ . The Galois group $\operatorname{Gal}(K/K^+)$ acts on X_{∞} . Let τ be the generator of $\operatorname{Gal}(K/K^+)$ and set

$$X_{\infty}^{-} := \{ x \in X_{\infty} \mid \tau(x) = -x \}.$$

Then, X_{∞}^- is a Λ -module whose μ -invariant (resp. λ -invariant) is denoted $\mu_p^-(K)$ (resp. $\lambda_p^-(K)$). Let $f_K^-(T)$ be the characteristic element associated to X_{∞}^- . Take I to be the set of primes of K^+ that lie above p and split in K, and set $r_{p,K} := \#I$. Let S be the set of places of K dividing p and ∞ . Let $U = U_K$ be the group of S-units of K^* and $M = M_K$ be the free abelian group of divisors at S. Given a $\operatorname{Gal}(K/K^+)$ -module N, let

$$N^{-} := \{ n \in N \mid \tau(n) = -n \}.$$

Let R be a ring, then we set RN^- to denote the extension of scalars $N^- \otimes R$. It is easy to see that both U^- and M^- are free abelian group of rank $r_{p,K}$.

The map

$$\phi: U^- \to M^-$$

is defined by setting

$$\phi(x) := \sum_{v|p} \operatorname{ord}_p \left(\operatorname{Norm}_{K_v/\mathbb{Q}_p} x \right).$$

The induced map

$$\phi: \mathbb{Q}U^- \to \mathbb{Q}M^-$$

is an isomorphism (cf. [FG80, Proposition 1.4]). The inverse of ϕ can be described as follows. For each prime $v \in I$, we choose a prime $\tilde{v}|v$ of K. Note that $\mathbb{Q}M^-$ has a basis

$$\mathcal{B} = \{ \widetilde{v} - \tau(\widetilde{v}) \mid v \in I \}.$$

Let \mathfrak{p} be the prime ideal associated to v, and h be a positive integer such that $\mathfrak{p}^h = (\alpha)$ is principal. Both α and $\tau(\alpha)$ are elements of U, and $\alpha/\tau(\alpha) \in U^-$. Setting f_v to denote the residue class degree of v, take

$$\phi^{-1}\left(\widetilde{v}-\tau(\widetilde{v})\right) := \frac{1}{hf_v} \otimes \left(\alpha/\tau(\alpha)\right).$$

Then, ϕ^{-1} is the inverse to ϕ .

Define the homomorphism

$$\lambda: U^- \to \mathbb{Q}_p M^-$$

by setting

$$\lambda(y) := \sum_{v|p} \log_p \left(N_{K_v/\mathbb{Q}_p}(y) \right).$$

Composing ϕ^{-1} with λ , we obtain an endomorphism

$$\lambda \phi^{-1} : \mathbb{Q}_p M^- \to \mathbb{Q}_p M^-.$$

Definition 2.4. With notation as above, define the regulator $\operatorname{Reg}_p(K)$ as follows

$$\operatorname{Reg}_p(K) := \det\left(\lambda\phi^{-1} \mid \mathbb{Q}_p M^-\right)$$

where it is understood that $\operatorname{Reg}_p(K) := 1$ when $r_{p,K} = 0$.

Theorem 2.5 (Iwasawa, Greenberg). With respect to notation above, $f_K^-(T)$ is divisible by $T^{r_{p,K}}$.

Proof. We refer to works of Iwasawa [Iwa73] and Greenberg [Gre73] for the proof of the statement. \Box

As a consequence of Theorem 2.5, one may write

$$f_K^-(T) = a_r T^r + a_{r+1} T^{r+2} + \dots + a_\lambda T^\lambda,$$

where $r = r_{p,K}$ and $\lambda := \lambda_p^-(K)$. We note that as a consequence,

$$\lambda_p^-(K) \ge r_{p,K}.$$

For $a, b \in \mathbb{Q}_p$, we write $a \sim b$ to mean that a = ub for some $u \in \mathbb{Z}_p^{\times}$. Let h_K (resp. h_K^-) denote the class number (resp. number of elements in the minus part of the class group) of K. Let $w_{K(\mu_p)}$ be the number of roots of unity contained in $K(\mu_p)$.

Theorem 2.6 (Federer-Gross). With respect to notation above, the following are equivalent

(1) $\operatorname{Reg}_p(K) \neq 0$, (2) $a_r \neq 0$.

Assuming that these equivalent conditions hold, we have that

$$a_r \sim \frac{h_K^- \left(\prod_{v \in I} f_v\right) \operatorname{Reg}_p(K)}{w_{K(\mu_p)}^{r_{p,K}}}$$

Proof. The above result is [FG80, Proposition 3.9].

Suppose that $p \nmid f_v$ for all $v \in I$, and $p \nmid h_K$. Then, the map ϕ^{-1} is defined over \mathbb{Z}_p and it is easy to see that $\operatorname{Reg}_p(K)$ is divisible by $p^{r_{p,K}}$. The normalized regulator is defined as follows

$$\mathcal{R}_p(K) := \frac{\operatorname{Reg}_p(K)}{p^{r_{p,K}}}.$$

Corollary 2.7. Let K be a CM field for which

- (i) $\mathcal{R}_{p}(K) \neq 0$, (ii) $w_{K(\mu_{p})} \sim p$, (iii) $p \nmid f_{v}$ for all $v \in I$, (iv) $p \nmid h_{K}$. Then, the following are equivalent
 - (1) $\mu_p^-(K) = 0$ and $\lambda_p^-(K) = r_{p,K}$, (2) a_r is a unit in \mathbb{Z}_p , (3) $p \nmid \mathcal{R}_p(K)$.

Proof. With respect to above notation, write $f_{K}^{-}(T) = T^{r_{p,K}}g_{K}^{-}(T)$, where $a_{r} = g_{K}^{-}(0)$. Then, a_{r} is a unit in \mathbb{Z}_{p} if and only if $g_{K}^{-}(T)$ is a unit in Λ . This implies that

$$\mu_p^-(K) = 0$$
 and $\lambda_p^-(K) = r_{p,K}$

Conversely, if

$$\mu_p^-(K) = 0$$
 and $\lambda_p^-(K) = r_{p,K}$

then the factorization $f_K^-(T) = T^{r_{p,K}} g_K^-(T)$ implies that $g_K^-(T)$ must be a unit in Λ . Thus, we find that (1) and (2) are equivalent. It follows from our assumptions that $a_r \sim \mathcal{R}_p(K)$. Therefore, a_r is a unit in \mathbb{Z}_p if and only if $p \nmid \mathcal{R}_p(K)$. This shows that the conditions (2) and (3) are equivalent.

When K is an imaginary quadratic field, $X_{\infty} = X_{\infty}^{-}$ and thus $\mu_p(K) = \mu_p^{-}(K)$ and $\lambda_p(K) = \lambda_p^{-}(K)$. In this setting, we find that

$$r_{p,K} := \begin{cases} 0 \text{ if } p \text{ splits in } K; \\ 1 \text{ if } p \text{ is inert or ramified in } K. \end{cases}$$

Note that $\mu_p(K) = 0$ by the aforementioned result of Ferrero and Washington [FW79].

Corollary 2.8. Let K be an imaginary quadratic field and p be an odd prime number. Then, the following assertions hold.

- (1) Suppose that p splits in K. Then, $\lambda_p(K) = 0$ if and only if $p \nmid h_K$.
- (2) Suppose that p is inert in K. Then, $\lambda_p(K) = 1$ if and only if $p \nmid h_K$ and $p \nmid \mathcal{R}_p(K/\mathbb{Q})$.

Proof. The result is a special case of Corollary 2.7.

Let K be an imaginary quadratic field and p be an odd prime which is inert in K. In this case $\lambda_p(K) \geq 1$ and the Corollary 2.8 asserts that $\lambda_p(K) = 1$ if and only if $\mathcal{R}_p(K/\mathbb{Q})$ is a unit in \mathbb{Z}_p . The analysis of this condition leads to the statement of Gold's criterion.

Theorem 2.9 (Gold's criterion). Let K be an imaginary quadratic field and p be an odd prime number which splits in K. Assume that $p \nmid h_K$. Let $\mathfrak{p}|p$ be a prime ideal, and r be a positive integer not divisible by p, such that \mathfrak{p}^r is principal. Let $\alpha \in \mathcal{O}_K$ be a generator of \mathfrak{p}^r . Setting $\bar{\mathfrak{p}}$ to denote the complex conjugate of \mathfrak{p} , the following conditions are equivalent

(1)
$$\lambda_p(K) > 1$$
,
(2) $\alpha^{p-1} \equiv 1 \pmod{\bar{\mathfrak{p}}^2}$.

Proof. The result is a consequence of [Gol74, Theorems 3 and 4], and can also be seen to follow from Corollary 2.8, cf. [San93, proof of Proposition 2.1]. \Box

2.3. Artin representations. We briefly discuss the results of Greenberg and Vatsal [GV20] on the Iwasawa theory of Selmer groups associated to Artin representations. Let K/\mathbb{Q} be a finite Galois extension with Galois group $\Delta := \text{Gal}(K/\mathbb{Q})$. Fix an irreducible Artin representation of dimension d > 1

$$\rho: \Delta \to \mathrm{GL}_d(\mathbb{Q})$$

and

$$S_{\chi,\epsilon}(\mathbb{Q}_{\infty}) := \varinjlim_n S_{\chi,\epsilon}(\mathbb{Q}_n)$$

the Selmer group over the cyclotomic \mathbb{Z}_p -extension of \mathbb{Q} . Note that $S_{\chi,\epsilon}(\mathbb{Q}_n)$ (resp. $S_{\chi,\epsilon}(\mathbb{Q}_\infty)$) is an $\mathcal{O}[\Gamma/\Gamma_n]$ -module (resp. $\Lambda_{\mathcal{O}}$ -module).

It is easy to see that the restriction map

$$H^1(\mathbb{Q}_n, D) \to H^1(\mathbb{Q}_\infty, D)^{\Gamma_n}$$

induces a map between Selmer groups

$$\iota_n: S_{\chi,\epsilon}(\mathbb{Q}_n) \to S_{\chi,\epsilon}(\mathbb{Q}_\infty)^{\Gamma_n}$$

The following control theorem shows that this map is injective with cokernel which is independent of n.

Theorem 2.10 (Greenberg and Vatsal – Control theorem). Suppose that χ is nontrivial, then, ι_n fits into a short exact sequence of Γ/Γ_n -modules

$$0 \to S_{\chi,\epsilon}(\mathbb{Q}_n) \to S_{\chi,\epsilon}(\mathbb{Q}_\infty)^{\Gamma_n} \to H^0(\Delta_{\mathfrak{p}}, D/D^{\epsilon_{\mathfrak{p}}}) \to 0.$$

Proof. The above result is [GV20, Proposition 4.1].

We note that $H^0(\Delta_{\mathfrak{p}}, D/D^{\epsilon_{\mathfrak{p}}}) \simeq (\mathcal{F}/\mathcal{O})^t$, where t is the multiplicity of the trivial representation of $\Delta_{\mathfrak{p}}$ in $V/V^{\epsilon_{\mathfrak{p}}}$. Here, the action of Γ/Γ_n on $H^0(\Delta_{\mathfrak{p}}, D/D^{\epsilon_{\mathfrak{p}}})$ is trivial.

Let \mathcal{D} be a discrete *p*-primary $\Lambda_{\mathcal{O}}$ -module. We take \mathcal{D}^{\vee} to be the Pontryagin dual defined as follows

$$\mathcal{D}^ee \coloneqq \operatorname{Hom}_{\mathbb{Z}_p}\left(\mathcal{D}, \mathbb{Q}_p/\mathbb{Z}_p
ight)$$
 .

Say that \mathcal{D} is cofinitely generated (resp. cotorsion) as a $\Lambda_{\mathcal{O}}$ -module, if \mathcal{D}^{\vee} is finitely generated (resp. torsion) as a $\Lambda_{\mathcal{O}}$ -module. We set $X_{\chi,\epsilon}(\mathbb{Q}_{\infty})$ be the Pontryagin dual of $S_{\chi,\epsilon}(\mathbb{Q}_{\infty})$.

Theorem 2.11 (Greenberg-Vatsal). With respect to notation above $X_{\chi,\epsilon}(\mathbb{Q}_{\infty})$ is a finitely generated and torsion $\Lambda_{\mathcal{O}}$ -module that contains no non-trivial finite $\Lambda_{\mathcal{O}}$ -submodules.

Proof. We refer to [GV20, Propositions 4.5 and 4.7] for the proof of the above result. \Box

Set $\mu_{\chi,\epsilon}$ and $\lambda_{\chi,\epsilon}$ to denote the μ and λ -invariants of $S_{\chi,\epsilon}(\mathbb{Q}_{\infty})^{\vee}$ respectively. The characteristic series of $S_{\chi,\epsilon}(\mathbb{Q}_{\infty})^{\vee}$ is denoted $f_{\chi,\epsilon}(T)$.

3. The Euler characteristic

In this section, we introduce the notion of the Euler characteristic associated to a cofinitely generated and cotorsion module over $\Lambda_{\mathcal{O}}$, and introduce conditions for it to be well defined. For the Selmer group associated to an Artin representation, the Euler characteristic is shown to equal the cardinality of $S_{\chi,\epsilon}(\mathbb{Q})$. The results presented in this section have consequences to the study of the Iwasawa μ and λ -invariants associated to an Artin representation.

3.1. Definition and properties of the Euler characteristic. Let \mathcal{D} be a cofinitely generated and cotorsion module over $\Lambda_{\mathcal{O}}$. Since Γ is pro-cyclic, it has cohomological dimension 1, we find that $H^i(\Gamma, \mathcal{D}) = 0$ for $i \geq 2$. Also note that $H^1(\Gamma, \mathcal{D})$ is identified with the module of coinvariants \mathcal{D}_{Γ} . This invariant encodes information about the valuation of leading constant term of the characteristic series. For an Artin representation that satisfies the conditions of Assumption 1.1 we discuss conditions for the Euler characteristic to be well defined, and give an explicit formula for it. The Euler characteristic formula in this context can be viewed as an analogue of the result of Gross and Federer.

Proposition 3.1. Let \mathcal{D} be a cofinitely generated and cotorsion $\Lambda_{\mathcal{O}}$ -module, then,

- (1) $\operatorname{corank}_{\mathbb{Z}_p} H^0(\Gamma, \mathcal{D}) = \operatorname{corank}_{\mathbb{Z}_p} H^1(\Gamma, \mathcal{D}).$
- (2) The module $H^0(\Gamma, \mathcal{D})$ is finite if and only if $H^1(\Gamma, \mathcal{D})$ is finite.

Proof. Since \mathcal{D} is cofinitely generated as a $\Lambda_{\mathcal{O}}$ -module, it follows that both \mathcal{D}^{Γ} and \mathcal{D}_{Γ} are cofinitely generated as a \mathbb{Z}_p -module. Part (1) follows from [How02, Theorem 1.1]. Since \mathcal{D} is cofinitely generated as a Λ -module, it follows that \mathcal{D}^{Γ} and \mathcal{D}_{Γ} are both cofinitely generated as \mathbb{Z}_p -modules. Part (2) follows from (1).

Definition 3.2. Let \mathcal{D} be a cofinitely generated and cotorsion $\Lambda_{\mathcal{O}}$ -module. Then, we say that the Euler characteristic of \mathcal{D} is well defined if $H^0(\Gamma, \mathcal{D})$ (or equivalently $H^1(\Gamma, \mathcal{D})$) is finite. Assuming that the characteristic of \mathcal{D} is well defined, the Euler characteristic is defined as follows

$$\chi(\Gamma, \mathcal{D}) := \prod_{i \ge 0} \# H^i(\Gamma, \mathcal{D})^{(-1)^i} = \frac{\# H^0(\Gamma, \mathcal{D})}{\# H^1(\Gamma, \mathcal{D})}$$

Recall from Definition 2.1 that $f_{\mathcal{D}^{\vee}}(T)$ is the characteristic element of \mathcal{D}^{\vee} and that $\mu_p(\mathcal{D}^{\vee})$ (resp. $\lambda_p(\mathcal{D}^{\vee})$) is the μ -invariant (resp. λ -invariant). For the ease of notation, we set

$$f_{\mathcal{D}}(T) := f_{\mathcal{D}^{\vee}}(T), \mu_p(\mathcal{D}) := \mu_p(\mathcal{D}^{\vee}) \text{ and } \lambda_p(\mathcal{D}) := \lambda_p(\mathcal{D}^{\vee}).$$

From the expansion

$$f_{\mathcal{D}}(T) = \sum_{i=0}^{\lambda} a_i T^i,$$

we note that $\lambda = \lambda_p(\mathcal{D})$. Moreover, $\mu_p(\mathcal{D}) = 0$ if and only if $\varpi \nmid a_i$ for at least one coefficient a_i . In particular, we have the have that $\varpi \nmid f_{\mathcal{D}}(0)$ if and only if $\mu_p(\mathcal{D}) = \lambda_p(\mathcal{D}) = 0$. Fix an absolute value $|\cdot|_{\varpi}$ on \mathcal{O} normalized by setting

$$|\varpi|_{\varpi} := [\mathcal{O} : \varpi \mathcal{O}]^{-1}$$

Proposition 3.3. Let \mathcal{D} be a cofinitely generated and cotorsion Λ -module. Then, with respect to the above notation, the following conditions are equivalent

(1) $f_{\mathcal{D}}(0) \neq 0$,

(2) the Euler characteristic $\chi(\Gamma, \mathcal{D})$ is well defined.

Moreover, if the above equivalent conditions are satisfied, then,

$$\chi(\Gamma, \mathcal{D}) = |f_{\mathcal{D}}(0)|_{\varpi}^{-1}.$$

Proof. It is easy to see that if \mathcal{D} and \mathcal{D}' are cofinitely generated $\Lambda_{\mathcal{O}}$ -modules that are pseudo-isomorphic, then \mathcal{D}^{Γ} is finite if and only if $(\mathcal{D}')^{\Gamma}$ is finite. Therefore, $\chi(\Gamma, \mathcal{D})$ is well defined if and only if $\chi(\Gamma, \mathcal{D}')$ is well defined. The characteristic series is determined up to pseudo-isomorphism, and therefore, $f_{\mathcal{D}}(T) = f_{\mathcal{D}'}(T)$. We assume without loss of generality that

$$\mathcal{D}^{\vee} = \left(\bigoplus_{i=1}^{s} \frac{\mathcal{O}\llbracket T \rrbracket}{(\varpi^{\mu_{i}})} \right) \oplus \left(\bigoplus_{j=1}^{t} \frac{\mathcal{O}\llbracket T \rrbracket}{(f_{i}(T)^{\lambda_{i}})} \right).$$

We identify $\left(\mathcal{D}^{\Gamma}\right)^{\vee}$ with

$$(\mathcal{D}^{\vee})_{\Gamma} = \mathcal{D}^{\vee}/T\mathcal{D}^{\vee} \simeq \left(\bigoplus_{i=1}^{s} \mathcal{O}/(\varpi^{\mu_{i}})\right) \oplus \left(\bigoplus_{j=1}^{t} \mathcal{O}/(f_{i}(0)^{\lambda_{i}})\right).$$

Since

$$f_{\mathcal{D}}(0) = \prod_{i} \varpi^{\mu_{i}} \times \prod_{j} f_{j}(0)^{\lambda_{j}},$$

we deduce that

$$#H^0(\Gamma, \mathcal{D}) = #H^1(\Gamma, \mathcal{D}^{\vee}) = #(\mathcal{D}^{\vee})_{\Gamma}$$

is finite if and only if $f_{\mathcal{D}}(0)$ is non-zero. Therefore, the Euler characteristic is well defined if and only if $f_{\mathcal{D}}(0)$ is non-zero.

Assuming that $f_{\mathcal{D}}(0) \neq 0$, we calculate the Euler characteristic. First, we note that the previous argument implies that

$$#H^0(\Gamma, \mathcal{D}) = |f_{\mathcal{D}}(0)|_{\varpi}^{-1}.$$

On the other hand, we find that

$$#H^1(\Gamma, D) = #H^0(\Gamma, D^{\vee}) = #(D^{\vee})^{\Gamma}$$

Identify $(\mathcal{D}^{\vee})^{\Gamma}$ with the kernel of the multiplication by T endomorphism of \mathcal{D}^{\vee} . Since $f_{\mathcal{D}}(0)$ is assumed to be non-zero, it follows that none of the terms $f_j(T)$ are divisible by T. It follows from this that the multiplication by T map is injective and hence, $(\mathcal{D}^{\vee})^{\Gamma} = 0$. Thus it has been shown that

$$\chi(\Gamma, D) = |f_{\mathcal{D}}(0)|_{\varpi}^{-1}.$$

Corollary 3.4. Suppose that the Euler characteristic of \mathcal{D} is well defined. Then the following conditions are equivalent

(1) $\chi(\Gamma, \mathcal{D}) = 1,$ (2) $\mu_p(\mathcal{D}) = 0 \text{ and } \lambda_p(\mathcal{D}) = 0.$

Proof. It is easy to see that $\mu_p(\mathcal{D}) = 0$ and $\lambda_p(\mathcal{D}) = 0$ if and only if $f_{\mathcal{D}}(T)$ is a unit in $\Lambda_{\mathcal{O}}$. Thus, the condition (2) is equivalent to the condition that $\varpi \nmid f_{\mathcal{D}}(0)$. According to Proposition 3.3,

$$\chi(\Gamma, \mathcal{D}) = |f_{\mathcal{D}}(0)|_{\varpi}^{-1}.$$

Therefore $\chi(\Gamma, \mathcal{D}) = 1$ if and only if $f_{\mathcal{D}}(0)$ is a unit in \mathcal{O} .

3.2. Calculating the Euler characteristic. Let ρ be an Artin representation with character χ and set ϵ to be the chosen character with respect to which the Selmer group $S_{\chi,\epsilon}(\mathbb{Q}_{\infty})$ is defined. We shall assume throughout that the Assumption 1.1 is satisfied.

Proposition 3.5. With respect to above notation, the following conditions are equivalent

(1) the Euler characteristic $\chi(\Gamma, S_{\chi,\epsilon}(\mathbb{Q}_{\infty}))$ is defined, (2) $H^0(\Delta_{\mathfrak{p}}, D/D^{\epsilon_{\mathfrak{p}}}) = 0.$

Proof. The Euler characteristic is defined if and only if $S_{\chi,\epsilon}(\mathbb{Q}_{\infty})^{\Gamma}$ is finite. By the control theorem, we have the short exact sequence

$$0 \to S_{\chi,\epsilon}(\mathbb{Q}) \to S_{\chi,\epsilon}(\mathbb{Q}_{\infty})^{\Gamma} \to H^0(\Delta_{\mathfrak{p}}, D/D^{\epsilon_{\mathfrak{p}}}) \to 0,$$

where $H^0(\Delta_{\mathfrak{p}}, D/D^{\epsilon_{\mathfrak{p}}})$ is a cofree \mathcal{O} -module. According to Theorem 1.2, the Selmer group $S_{\chi,\epsilon}(\mathbb{Q})$ is finite. Therefore, the Euler characteristic is well defined if and only if $H^0(\Delta_{\mathfrak{p}}, D/D^{\epsilon_{\mathfrak{p}}}) = 0.$

Theorem 3.6. Let ρ be an Artin representation for which

the conditions of Assumption 1.1 are satisfied.
 H⁰(Δ_p, D/D^{ε_p}) = 0.

Then, the Euler characteristic is given by

$$\chi(\Gamma, S_{\chi,\epsilon}(\mathbb{Q}_{\infty})) = \#S_{\chi,\epsilon}(\mathbb{Q})$$

Proof. According to Proposition 3.5, the Euler characteristic is well defined. We are to calculate the orders of the finite abelian groups $S_{\chi,\epsilon}(\mathbb{Q}_{\infty})^{\Gamma}$ and $S_{\chi,\epsilon}(\mathbb{Q}_{\infty})_{\Gamma}$. According to Theorem 2.11, $X_{\chi,\epsilon}(\mathbb{Q}_{\infty}) := S_{\chi,\epsilon}(\mathbb{Q}_{\infty})^{\vee}$ does not contain any nontrivial finite $\Lambda_{\mathcal{O}}$ -submodules. Since the Euler characteristic is well defined, $X_{\chi,\epsilon}(\mathbb{Q}_{\infty})^{\Gamma}$ is finite, and hence is zero. In other words, $S_{\chi,\epsilon}(\mathbb{Q}_{\infty})_{\Gamma} = 0$. On the other hand, it follows from Theorem 2.10 that

$$S_{\chi,\epsilon}(\mathbb{Q}_{\infty})^{\Gamma} = S_{\chi,\epsilon}(\mathbb{Q}).$$

Therefore, we find that $\chi(\Gamma, S_{\chi,\epsilon}(\mathbb{Q}_{\infty})) = \#S_{\chi,\epsilon}(\mathbb{Q}).$

Corollary 3.7. Let ρ be an Artin representation and $\epsilon : \operatorname{Gal}(\mathcal{K}/\mathbb{Q}_p) \to \mathcal{O}^{\times}$ be a character for which the following assumptions are satisfied.

- (i) The conditions of Assumption 1.1 are satisfied.
- (*ii*) $H^0(\Delta_{\mathfrak{p}}, D/D^{\epsilon_{\mathfrak{p}}}) = 0.$

Then, the following conditions are equivalent

(1) $S_{\chi,\epsilon}(\mathbb{Q}_{\infty}) = 0,$ (2) $S_{\chi,\epsilon}(\mathbb{Q}_{\infty})$ is finite, (3) $\mu_{\chi,\epsilon} = \lambda_{\chi,\epsilon} = 0,$ (4) $\chi(\Gamma, S_{\chi,\epsilon}(\mathbb{Q}_{\infty})) = 1,$ (5) $S_{\chi,\epsilon}(\mathbb{Q}) = 0.$

Proof. It is clear that (1) implies (2). On the other hand, by Theorem 2.11, $X_{\chi,\epsilon}(\mathbb{Q}_{\infty})$ contains no nontrivial finite $\Lambda_{\mathcal{O}}$ -submodules. Hence, if $S_{\chi,\epsilon}(\mathbb{Q}_{\infty})$ is finite, it must be equal to 0. Therefore, (1) and (2) are equivalent.

We note that a $\Lambda_{\mathcal{O}}$ -module M is finite if and only if it is pseudo-isomorphic to the trivial module. On the other hand, $M \sim 0$ if and only if the μ and λ -invariants of M are 0. Therefore (2) and (3) are equivalent.

The equivalence of (3) and (4) follows from Corollary 3.4, and the equivalence of (4) and (5) follows from Proposition 3.3.

3.3. The structure of the Selmer group $S_{\chi,\epsilon}(\mathbb{Q})$. In this subsection, we analyze the structure of $S_{\chi,\epsilon}(\mathbb{Q})$ and establish conditions for the vanishing of $S_{\chi,\epsilon}(\mathbb{Q}_{\infty})$. We set $d(\chi)$ to denote d and write $d(\chi) = d^+(\chi) + d^-(\chi)$, reflecting the action of the archimedian places on the representation. At each prime $\mathfrak{p}|p$, let $\mathcal{U}_{\mathfrak{p}}$ denote the principal units at \mathfrak{p} , and set \mathcal{U}_p to denote the product $\prod_{\mathfrak{p}|p} \mathcal{U}_{\mathfrak{p}}$. The decomposition group $\Delta_{\mathfrak{p}}$ naturally acts on $\mathcal{U}_{\mathfrak{p}}$, and \mathcal{U}_p is identified with the induced representation $\mathrm{Ind}_{\Delta_{\mathfrak{p}}}^{\Delta} \mathcal{U}_{\mathfrak{p}}$. Let U_K be the group of units of \mathcal{O}_K that are principal units at all primes $\mathfrak{p}|p$. The diagonal inclusion map

$$U_K \hookrightarrow \mathcal{U}_p$$

is a Δ -equivariant map, and induces a \mathbb{Z}_p -linear map

$$\lambda_p: U_K \otimes \mathbb{Z}_p \to \mathcal{U}_p.$$

We recall that the character ϵ arises from a choice of prime $\mathfrak{p}|p$ and character $\epsilon_{\mathfrak{p}} : \Delta_{\mathfrak{p}} \to \mathcal{O}^{\times}$. Let \mathfrak{p}' be any other prime above p, and write $\mathfrak{p}' = \delta(\mathfrak{p})$ for $\delta \in \Delta$. Then conjugation by δ gives an isomorphism $c_{\delta} : \Delta_{\mathfrak{p}'} \xrightarrow{\sim} \Delta_{\mathfrak{p}}$. We set $\epsilon_{\mathfrak{p}'}$ to denote the composite

$$\Delta_{\mathfrak{p}'} \xrightarrow{c_{\delta}} \Delta_{\mathfrak{p}} \xrightarrow{\epsilon_{\mathfrak{p}}} \mathcal{O}^{\times}$$

Thus for each prime $\mathfrak{p}|p$, we have made a choice of character ϵ_p of $\Delta_{\mathfrak{p}}$. For $\mathfrak{p}|p$, we have a decomposition of $\mathcal{O}[\Delta_{\mathfrak{p}}]$ -modules,

$$\mathcal{U}_{\mathfrak{p},\mathcal{O}} = \mathcal{U}_{\mathfrak{p},\mathcal{O}}^{\epsilon_{\mathfrak{p}}} imes \mathcal{V}_{\mathfrak{p},\mathcal{O}},$$

where the action of $\Delta_{\mathfrak{p}}$ on $\mathcal{U}_{\mathfrak{p},\mathcal{O}}^{\epsilon_{\mathfrak{p}}}$ is via $\epsilon_{\mathfrak{p}}$. Use the following notation

$$\mathcal{U}_{p,\mathcal{O}}^{[\epsilon]} := \prod_{\mathfrak{p}|p} \mathcal{U}_{\mathfrak{p},\mathcal{O}}^{\epsilon_{\mathfrak{p}}}, \text{ and } \mathcal{V}_{p,\mathcal{O}} = \prod_{\mathfrak{p}|p} \mathcal{V}_{\mathfrak{p},\mathcal{O}}.$$

Then, we find that

$$\mathcal{U}_{p,\mathcal{O}} = \mathcal{U}_{p,\mathcal{O}}^{[\epsilon]} imes \mathcal{V}_{p,\mathcal{O}}.$$

The above decomposition is that of $\mathcal{O}[\Delta]$ -modules. In the same way, $\bar{U}_{p,\mathcal{O}}$ decomposes into a product

$$\bar{U}_{p,\mathcal{O}} = \bar{U}_{p,\mathcal{O}}^{[\epsilon]} \times \bar{V}_{p,\mathcal{O}}.$$

Let $\left(\bar{U}_{p,\mathcal{O}}^{[\epsilon]}\right)^{\chi}$ (resp. $\left(\bar{U}_{p,\mathcal{O}}^{[\epsilon]}\right)^{\chi}$) denote the χ -isotypic component of $\left(\bar{U}_{p,\mathcal{O}}^{[\epsilon]}\right)$ (resp. $\left(\bar{U}_{p,\mathcal{O}}^{[\epsilon]}\right)$). Greenberg and Vatsal show that $\left(\bar{U}_{p,\mathcal{O}}^{[\epsilon]}\right)^{\chi}$ has finite index in $\left(\mathcal{U}_{p,\mathcal{O}}^{[\epsilon]}\right)^{\chi}$, provided the assertions of Assumption 1.1 are satisfied.

Proposition 3.8. With respect to notation above, suppose that the Assumption 1.1 is satisfied. Then, the following assertions hold

(1) $\left(\bar{U}_{p,\mathcal{O}}^{[\epsilon]}\right)^{\chi}$ has finite index in $\left(\mathcal{U}_{p,\mathcal{O}}^{[\epsilon]}\right)^{\chi}$, (2) there is a short exact sequence of \mathcal{O} -modules

$$0 \to H^1_{\mathrm{nr}}(\mathbb{Q}, D) \to S_{\chi, \epsilon}(\mathbb{Q}) \to \mathrm{Hom}_{\mathcal{O}[\Delta]}\left(\left(\mathcal{U}_{p, \mathcal{O}}^{[\epsilon]}\right)^{\chi} / \left(\bar{U}_{p, \mathcal{O}}^{[\epsilon]}\right)^{\chi}, D\right) \to 0.$$

Here, $H^1_{nr}(\mathbb{Q}, D)$ is the subgroup of $H^1(\mathbb{Q}, D)$ consisting of cohomology classes that are unramified at all primes.

Proof. The result follows from the proof of [GV20, Proposition 3.1].

Theorem 3.9. Assume that

- (i) the conditions of Assumption 1.1 are satisfied.
- (*ii*) $H^0(\Delta_{\mathfrak{p}}, D/D^{\epsilon_{\mathfrak{p}}}) = 0.$

Then, the following conditions are equivalent

(1) $H^1_{\mathrm{nr}}(\mathbb{Q}, D) = 0$ and $\left(\mathcal{U}_{p,\mathcal{O}}^{[\epsilon]}\right)^{\chi} = \left(\bar{U}_{p,\mathcal{O}}^{[\epsilon]}\right)^{\chi}$. (2) $\mu_{\chi,\epsilon} = \lambda_{\chi,\epsilon} = 0$. (3) $S_{\chi,\epsilon}(\mathbb{Q}_{\infty}) = 0$,

Proof. The result follows as a direct consequence of Corollary 3.7 and Proposition 3.8.

Lemma 3.10. With respect to notation above, assume that $p \nmid \# \operatorname{Cl}(K)$. Then, we find that $H^1_{\operatorname{nr}}(\mathbb{Q}, D) = 0$.

Proof. Since $p \nmid \#\Delta$, we find that $H^1(\Delta, D^{G_K}) = 0$, and thus from the inflationrestriction sequence, $H^1(\mathbb{Q}, D)$ injects into $\operatorname{Hom}\left(\operatorname{G}_K^{\operatorname{ab}}, D\right)^{\Delta}$. Let H_K be the Hilbert class field of K. We find that $H^1_{\operatorname{nr}}(\mathbb{Q}, D)$ injects into $\operatorname{Hom}(\operatorname{Gal}(H_K/K), D)^{\Delta}$. Since $p \nmid \#\operatorname{Cl}(K)$, we find that

$$\operatorname{Hom}(\operatorname{Gal}(H_K/K), D) = 0,$$

and hence, $H^1_{nr}(\mathbb{Q}, D) = 0$.

Theorem 3.11. Assume that

- (i) the conditions of Assumption 1.1 are satisfied. (ii) $H^0(\Delta_{\mathfrak{p}}, D/D^{\epsilon_{\mathfrak{p}}}) = 0$,
- (iii) K is p-rational,
- (iii) I is p-fullow
- (iv) $p \nmid \# \operatorname{Cl}(K)$.

Then, $S_{\chi,\epsilon}(\mathbb{Q}_{\infty}) = 0.$

Proof. Since $p \nmid \# \operatorname{Cl}(K)$, it follows from Lemma 3.10 that $H^1_{\operatorname{nr}}(\mathbb{Q}, D) = 0$. Let G_p be the maximal pro-p extension of K which is unramified outside p. Since $p \nmid \# \operatorname{Cl}(K)$, we find that G_p^{ab} is isomorphic to $\mathcal{U}_p/\overline{\mathcal{U}}$. On the other hand, K is p-rational if and only if the Leopoldt conjecture is true at p and there is no p torsion in G_p^{ab} (cf. [MR20] or [MD90]). This implies that

$$\left(\mathcal{U}_{p,\mathcal{O}}^{[\epsilon]}\right)^{\chi} / \left(\bar{U}_{p,\mathcal{O}}^{[\epsilon]}\right)^{\chi} = 0$$

The results then follow from Theorem 3.9.

3.4. Special cases. In this subsection, we consider certain special cases. First, we consider some 2-dimensional Artin representations of dihedral type. Let (L,ζ) be a pair, where L is an imaginary quadratic extension of \mathbb{Q} and $\zeta : \mathcal{G}_L \to \overline{\mathbb{Q}}^{\times}$ is a character. Consider the 2-dimensional Artin representation

$$\rho = \rho_{(L,\zeta)} := \operatorname{Ind}_{\mathcal{G}_L}^{\mathcal{G}_\mathbb{Q}} \zeta.$$

When restricted to L, the representation ρ decomposes into a direct sum of characters

$$\rho_{|G_L} = \left(\begin{array}{c} \zeta \\ & \zeta' \end{array}\right)$$

Here, ζ' is given as follows

$$\zeta'(x) = \zeta(cxc^{-1}),$$

where c denotes the complex conjugation. We remark that the representation ρ is a self-dual representation of dihedral type if and only if $\zeta' = \zeta^{-1}$. Let p be an odd prime number which splits in L as a product $p\mathcal{O}_L = \mathfrak{pp}^*$. We set $\epsilon_{\mathfrak{p}} := \zeta_{|G_{\mathfrak{p}}}$ and note that $\epsilon_{\mathfrak{p}^*} = \zeta'_{|G_{\mathfrak{p}}}$. The field $K = L(\zeta, \zeta')$ is the extension of L generated by ζ and ζ' . Choose an extension \mathcal{F}/\mathbb{Q}_p containing $\mathbb{Q}_p(\chi, \epsilon_{\mathfrak{p}})$ and set \mathcal{O} to denote its valuation ring, and D the associated \mathcal{O} -divisible module. With respect to such a choice, we let $S_{\chi,\epsilon}(\mathbb{Q}_{\infty})$ be the associated the Selmer group over the cyclotomic \mathbb{Z}_p -extension.

Theorem 3.12. With respect to notation above, assume that the following conditions hold

(i) ρ is irreducible,

- (ii) the order of ζ is coprime to p,
- (iii) $\epsilon_{\mathfrak{p}}$ is nontrivial and $\epsilon_{\mathfrak{p}} \neq \epsilon_{\mathfrak{p}^*}$.

Then, the following conditions are equivalent

(1) $H^1_{\mathrm{nr}}(\mathbb{Q}, D) = 0$ and $\left(\mathcal{U}_{p,\mathcal{O}}^{[\epsilon]}\right)^{\chi} = \left(\bar{U}_{p,\mathcal{O}}^{[\epsilon]}\right)^{\chi}$. (2) $\mu_{\chi,\epsilon} = \lambda_{\chi,\epsilon} = 0$.

(3)
$$S_{\chi,\epsilon}(\mathbb{Q}_{\infty}) = 0.$$

If $p \nmid h_K$ and K is p-rational, then, the above conditions are satisfied.

Proof. The result follows from Theorem 3.9 once we show that

- (i) the conditions of Assumption 1.1 are satisfied.
- (ii) $H^0(\Delta_{\mathfrak{p}}, D/D^{\epsilon_{\mathfrak{p}}}) = 0.$

Consider the conditions for Assumption 1.1 to hold.

- (1) The field K is the extension of L cut out by ζ and ζ' . Since the order of ζ is assumed to be coprime to p, the same holds for ζ' . Since p is odd, it follows that $p \nmid [K : \mathbb{Q}]$.
- (2) Since K is an imaginary quadratic field complex conjugation acts via the matrix $\begin{pmatrix} 1\\ 1 \end{pmatrix}$, whose eigenvalues are 1 and -1 respectively. Therefore, $d^+ = d^- = 1$.

(3) The third condition follows from the assumption that $\epsilon_{\mathfrak{p}}$ is nontrivial and $\epsilon_{\mathfrak{p}} \neq \epsilon_{\overline{\mathfrak{p}}}$. Finally, we note that since $\epsilon_{\overline{\mathfrak{p}}}$ is nontrivial, and hence $H^0(\Delta_{\mathfrak{p}}, D/D^{\epsilon_{\mathfrak{p}}}) = 0$.

By the proof of Theorem 3.11, we note that $H^1_{\mathrm{nr}}(\mathbb{Q}, D) = 0$ and $\left(\mathcal{U}_{p,\mathcal{O}}^{[\epsilon]}\right)^{\chi} = \left(\bar{U}_{p,\mathcal{O}}^{[\epsilon]}\right)^{\chi}$ if p is such that $p \nmid h_K$ and K is p-rational, then, the above conditions are satisfied. \Box

Next, we consider the group of rotational symmetries of the icosahedron. This group is simply A_5 . An orthogonal set is a set of 6 points, so that three pairwise orthogonal lines can be drawn between pairs of them. The midpoints of the 30 edges of an icosahedron can be partitioned into 5 orthogonal sets such that these sets are permuted by A_5 . Let $g \in A_5$ be an element that corresponds to rotation of the icosahedron with axis (x, y, z)and angle θ . Then the corresponding matrix is

$$\begin{pmatrix} \cos\theta + (1 - \cos\theta)x^2 & (1 - \cos\theta)xy - z\sin\theta & (1 - \cos\theta)xz + y\sin\theta \\ (1 - \cos\theta)xy + z\sin\theta & \cos\theta + (1 - \cos\theta)x^2 & (1 - \cos\theta)yz - x\sin\theta \\ (1 - \cos\theta)xz - y\sin\theta & (1 - \cos\theta)yz + x\sin\theta & \cos\theta + (1 - \cos\theta)z^2. \end{pmatrix}$$

This gives a representation

$$r: A_5 \to \mathrm{GL}_3(\bar{\mathbb{Q}}).$$

Let K/\mathbb{Q} be a Galois extension with Galois group $\operatorname{Gal}(K/\mathbb{Q})$ isomorphic to A_5 , and ρ be the composite

$$\varrho: \mathcal{G}_{\mathbb{Q}} \to \mathrm{Gal}(K/\mathbb{Q}) \xrightarrow{\sim} A_5 \xrightarrow{r} \mathrm{GL}_3(\bar{\mathbb{Q}}),$$

where the initial map is the quotient map. Any 5-cycle corresponds to a rotation by an angle of $\frac{2\pi}{5}$, and therefore has eigenvalues $e^{\frac{2\pi i}{5}}$, $e^{-\frac{2\pi i}{5}}$, 1. Let $g \in A_5$ be a 5-cycle and D be the group generated by g, and $L := K^D$ be the field fixed by D. Let $p \ge 7$ be a prime which is split in L and is inert in K/L. Then, the restriction to the decomposition group

at p is of the form diag $(\alpha_p, \alpha_p^{-1}, 1)$, where α_p is an unramified character of order 5. Let $\psi : G_{\mathbb{Q}} \to \mathcal{O}^{\times}$ be an even character which is ramified at p, and set $\rho := \rho \otimes \psi$.

Theorem 3.13. With respect to the above notation, the following conditions are equivalent

(1) $H^1_{\mathrm{nr}}(\mathbb{Q}, D) = 0$ and $\left(\mathcal{U}_{p,\mathcal{O}}^{[\epsilon]}\right)^{\chi} = \left(\bar{U}_{p,\mathcal{O}}^{[\epsilon]}\right)^{\chi}$. (2) $\mu_{\chi,\epsilon} = \lambda_{\chi,\epsilon} = 0$. (3) $S_{\chi,\epsilon}(\mathbb{Q}_{\infty}) = 0$.

If $p \nmid h_K$ and K is p-rational, then, the above conditions are satisfied.

Proof. As in the proof of Theorem 3.12, the result follows from Theorem 3.9 once we show that

- (i) the conditions of Assumption 1.1 are satisfied.
- (ii) $H^0(\Delta_{\mathfrak{p}}, D/D^{\epsilon_{\mathfrak{p}}}) = 0.$

Let us verify the conditions of the Assumption 1.1.

- (1) Since it is assumed that $p \ge 7$, it follows that $p \nmid \#A_5$, and hence, $p \nmid [K : \mathbb{Q}]$.
- (2) Let v be an archidean prime, and σ_v be a generator of the decomposition group at v. Then, $\rho(\sigma_v) = \rho(\sigma_v)\psi(\sigma_v) = \rho(\sigma_v)$. Since ρ only gives rise to rotational symmetries and $\rho(\sigma_v)$ has order 2, it must correspond to the rotation with $\theta = \pi$. This means that it is conjugate to diag(1, -1, -1), and hence, $d^+ = 1$.
- (3) For the third condition, it has been arranged that $\varrho_{|\Delta_{\mathfrak{p}}} = \operatorname{diag}(\alpha_p, \alpha_p^{-1}, 1)$, where α_p is an unramified character of order 5, and hence, $\epsilon_p := \alpha_p \psi$ is a ramified character which occurs with multiplicity 1.

Finally, we note that $H^0(\Delta_{\mathfrak{p}}, D/D^{\epsilon_p}) = 0$ since the characters $\alpha_p^{-1}\psi$ and ψ are nontrivial. This is because ψ is ramified at p, while α_p is not.

4. DISTRIBUTION QUESTIONS FOR ARTIN REPRESENTATIONS

4.1. Iwasawa theory of class groups. In this section, we introduce and study some natural distribution questions for Iwasawa invariants of class groups. These discussions serve to motivate similar questions for Artin representations, which we discuss in the next subsection. Given a number field K and a prime number p, the Iwasawa invariants $\mu_p(K)$ and $\lambda_p(K)$ are natural invariants to consider associated with the growth of p-primary parts of class groups in the cyclotomic extension of K. One considers the following question.

Question 4.1. Given an imaginary quadratic K/\mathbb{Q} , how does $\mu_p(K)$ and $\lambda_p(K)$ vary as $p \to \infty$? In other words, for a given pair (μ, λ) , what can be said about the upper and lower densities of the set of primes p for which

$$\lambda_p(K) = \lambda.$$

We call the above set \mathcal{F}_{λ} , and its upper (resp. lower) density $\bar{\mathfrak{o}}_{\lambda}$ (resp. $\underline{\mathfrak{o}}_{\lambda}$). We note that if p is inert or ramified in K and $p \nmid h_K$, then, $h_p(K_n) = 0$ for all n. In particular, $\lambda_p(K) = 0$. This implies in particular that $\underline{\mathfrak{o}}_0 \geq \frac{1}{2}$. On the other hand, if p splits in K, then, $r_{p,K} \geq 1$ and thus, $\lambda_p(K) \geq 1$. This implies that $\mathfrak{o}_0 = \frac{1}{2}$. Let p be a prime which splits in K for which $p \nmid h_K$. The former condition is satisfied by $\frac{1}{2}$ of the primes and the latter condition is satisfied for all but finitely many primes. The following result is a corollary to Gold's criterion.

Corollary 4.2. Let p be an odd prime number which splits into \mathfrak{pp}^* in \mathcal{O}_K . Suppose that r > 1 is an integer not divisible by p and such that $\mathfrak{p}^r = (\alpha)$. Then, the following conditions are equivalent

(1)
$$\lambda_p(K) > 1$$
,
(2) $\operatorname{Tr}(\alpha)^{p-1} \equiv 1 \mod p^2$

Indeed we find that for each number $a_0 \in [1, p-1]$, there is precisely one congruence class a modulo- p^2 such that is $a_0 \mod p$ and satisfies $a^{p-1} \equiv 1 \mod p^2$. Therefore, the probability that an integer $a \in \mathbb{Z}/p^2\mathbb{Z}$ satisfies the congruence $a^{p-1} \equiv 1 \mod p^2$ is $\frac{p-1}{p^2} = \frac{1}{p} - \frac{1}{p^2}$. Let $N_K(X)$ be the number of primes $p \leq X$ such that p is split in K and $\lambda_p(K) > 1$. Thus, the heuristic suggests that

$$N_K(X) \sim \sum_{p \in \Omega'(X)} \left(\frac{1}{p} - \frac{1}{p^2}\right) \sim \log \log X,$$

where $\Omega'(X)$ is the set of primes $p \leq X$ that split in K. This leads us to the following expectation.

Conjecture 4.3. Let $M_p(X)$ be the number of primes $\leq X$ that split in K for which $\lambda_p(K) > 1$, then, $M_p(X) = \frac{1}{2} \log \log X + O(1)$.

In particular the set of primes p for which $\lambda_p(K) > 1$ is infinite of density 0, and thus, the conjecture in particular predicts that $\mathfrak{d}_1 = \frac{1}{2}$. Horie [Hor87] has proven the infinitude of primes p such that $\lambda_p(K) > 1$ and Jochnowitz [Joc94] on the other hand the infinitude of primes p for which $\lambda_p(K) = 1$.

On the other hand, one can fix a prime and study the variation of $\lambda_p(K)$ as K ranges over all imaginary quadratic fields. We separate this problem into two cases, namely that of imaginary quadratic fields in which p is inert, and those in which p splits. For the primes p that are inert in K Ellenberg, Jain and Venkatesh [EJV11] make the following prediction based on random matrix heuristics.

Conjecture 4.4 (Ellenberg, Jain, Venkatesh). Amongst all imaginary quadratic fields K in which p is inert, the proportion for which $\lambda_p(K) = r$ is equal to

$$p^{-r}\prod_{t>r}\left(1-p^{-t}\right).$$

We note that $\lambda_p(K) = 0$ if and only if $p \nmid h_K$. The probability that $p \nmid h_K$ is, according to the Cohen-Lenstra heuristic, predicted to be equal to

$$\prod_{t>0} \left(1 - p^{-t}\right).$$

4.2. Artin representations. Given an Artin representation ρ , we are interested in understanding the variation of μ and λ -invariants as p ranges over all prime numbers. We specialize our discussion to odd 2-dimensional Artin representations of dihedral type

$$\rho = \operatorname{Ind}_{\mathbf{G}_L}^{\mathbf{G}_{\mathbb{Q}}} \zeta,$$

where L is an imaginary quadratic field and $\zeta : G_L \to \overline{\mathbb{Q}}^{\times}$ is a character. Let ζ' be the character defined by $\zeta'(\sigma) = \zeta(c\sigma c^{-1})$, where $c \in G_{\mathbb{Q}}$ denotes the complex conjugation. Here, L/\mathbb{Q} is an imaginary quadratic field. Since it is assumed that ρ is of dihedral type, it follows that $\zeta' = \zeta^{-1}$ and the extension K is simply the extension of L that is fixed by the kernel of ζ . Furthermore, assume that $2 \nmid [K : L]$. Let $S(\rho)$ be the set of primes p such that

- (1) p is odd and $p \nmid [K : \mathbb{Q}]$,
- (2) p splits in L, $p\mathcal{O}_L = \pi\pi^*$, and π is inert in K/L.

For $p \in S(\rho)$, it is then clear from the assumptions that the Assumption 1.1 holds, and that $H^0(\Delta_{\mathfrak{p}}, D/D^{\epsilon_{\mathfrak{p}}}) = 0$ for any of the primes $\mathfrak{p}|p$ of K and the character $\epsilon_{\mathfrak{p}} = \zeta_{|\Delta_{\mathfrak{p}}}$. Let ϵ be the character associated to this choice of \mathfrak{p} and $\epsilon_{\mathfrak{p}}$.

Definition 4.5. Let $T(\rho)$ be the set of primes $p \in S(\rho)$ such that the Selmer group $S_{\chi,\epsilon}(\mathbb{Q}_{\infty})$ vanishes for all of the primes \mathfrak{p} that lie above p. Set $T'(\rho) := S(\rho) \setminus T(\rho)$.

We note that the Theorem 3.11 implies that for $p \in S(\rho)$ such that K is p-rational, then, $p \in T(\rho)$.

Let us recall some heuristics on the *p*-rationality condition. The \mathbb{Z}_p -rank of \mathcal{U}_p is $n = [K : \mathbb{Q}] = r_1 + 2r_2$. On the other hand, the rank of U_K is $k := r_1 + r_2 - 1$. The probability that a random matrix with *n* columns and *k* rows does not have full \mathbb{Z}_p -rank is

$$\Pr_{k,n} := 1 - \frac{\prod_{i=0}^{k-1} \left(p^n - p^i \right)}{p^n} \le \frac{1}{p^{n-k+1}} + \frac{1}{p^{n-k+2}} + \dots + \frac{1}{p^n}.$$

Therefore, the expected number of primes p at which K is not p-rational is finite, since according to this heuristic,

$$\sum_{p} \Pr_{k,n} \le \sum_{p} \left(\frac{1}{p^{r_2+2}} + \frac{1}{p^{r_2+3}} + \dots + \frac{1}{p^n} \right) = \zeta(r_2+2) + \zeta(r_2+3) + \dots + \zeta(n) < \infty.$$

This is indeed a conjecture due to Gras.

Conjecture 4.6 (Gras). Let K be a number field. Then for all large values of p, K is p-rational.

This leads us to make the following conjecture.

Conjecture 4.7. With respect to above notation, the set of primes $T'(\rho)$ is finite. Thus, there are only finitely many pairs (\mathfrak{p}, ϵ) where \mathfrak{p} is a prime above $p \in S(\rho)$ and $\epsilon : G_{L_{\mathfrak{p}}} \to \mathbb{Q}_p^{\times}$ is a character, such that the associated Selmer group $S_{\chi,\epsilon}(\mathbb{Q}_{\infty}) \neq 0$.

There is a relationship between Gras' conjecture and the generalized abc-conjecture, stated below, cf. [Voj98].

Conjecture 4.8 (Generalized abc-conecture). Let K be a number field and I be an ideal in \mathcal{O}_K , the radical of I is defined as follows

$$\operatorname{Rad}(I) := \prod_{\mathfrak{p}|p} N(\mathfrak{p}),$$

where the product is over all prime ideals \mathfrak{p} dividing I, and $N(\mathfrak{p}) := \# (\mathcal{O}_K/\mathfrak{p}\mathcal{O}_K)$ is the norm of \mathfrak{p} . The generalized abc conjecture predicts that for any $\epsilon > 0$, there exists a constant $C_{K,\epsilon} > 0$ such that

$$\prod_{v} \max\{|a|_{v}, |b|_{v}, |c|_{v}\} \le C_{K,\epsilon} \left(\operatorname{Rad}(abc)\right)^{1+\epsilon},$$

holds for all non-zero $a, b, c \in \mathcal{O}_K$ such that a + b = c.

Let us recall a recent result of Maire and Rougnant, cf. [MR20, Theorem A].

Theorem 4.9 (Maire-Rougnant). Let K/\mathbb{Q} be an imaginary S_3 extension. Then, the generalized abc-conjecture for K implies that there is a constant c > 0 such that

$$\#\{p \le x \mid K \text{ is } p\text{-rational}\} \ge c \log x$$

The above result has implication to the vanishing of Iwasawa modules (and invariants).

Theorem 4.10. Let K/\mathbb{Q} be an imaginary S_3 extension and ρ be a 2-dimensional Artin representation that factors through $\operatorname{Gal}(K/\mathbb{Q})$. Then,

$$#\{p \le x \mid p \in T(\rho)\} \ge c \log x.$$

Proof. Theorem 3.11 implies that for $p \in S(\rho)$ such that K is p-rational, then, $p \in T(\rho)$. The result thus follows.

References

- [BDSBT01] Kevin Buzzard, Mark Dickinson, Nick Shepherd-Barron, and Richard Taylor. On icosahedral Artin representations. *Duke Math. J.*, 109(2):283–318, 2001.
- $\begin{array}{lll} \mbox{[EJV11]} & \mbox{Jordan S Ellenberg, Sonal Jain, and Akshay Venkatesh. Modeling λ-invariants by p-adic random matrices. Communications on pure and applied mathematics, 64(9):1243-1262, 2011. \end{array}$
- [FG80] Leslie Jane Federer and Benedict H Gross. Regulators and Iwasawa modules. Inventiones mathematicae, 62:443–457, 1980.
- [FW79] Bruce Ferrero and Lawrence C Washington. The Iwasawa invariant μ_p vanishes for abelian number fields. Annals of Mathematics, pages 377–395, 1979.
- [G⁺89] Ralph Greenberg et al. Iwasawa theory for p-adic representations. *Algebraic number theory*, 17:97–137, 1989.
- [Gol74] Robert Gold. The nontriviality of certain \mathbb{Z}_{ℓ} -extensions. Journal of Number Theory, 6(5):369–373, 1974.
- [Gre73] Ralph Greenberg. On a certain l-adic representation. *Inventiones mathematicae*, 21(1-2):117-124, 1973.
- [GV20] Ralph Greenberg and Vinayak Vatsal. Iwasawa theory for Artin representations I. In Development of Iwasawa theory—the centennial of K. Iwasawa's birth, volume 86 of Adv. Stud. Pure Math., pages 255–301. Math. Soc. Japan, Tokyo, [2020] ©2020.
- [HM19] Farshid Hajir and Christian Maire. Prime decomposition and the Iwasawa MU-invariant. Math. Proc. Cambridge Philos. Soc., 166(3):599–617, 2019.
- [Hor87] Kuniaki Horie. A note on basic Iwasawa λ -invariants of imaginary quadratic fields. Inventiones mathematicae, 88(1):31–38, 1987.
- [How02] Susan Howson. Euler characteristics as invariants of Iwasawa modules. Proceedings of the London Mathematical Society, 85(3):634–658, 2002.
- [Iwa59] Kenkichi Iwasawa. On γ -extensions of algebraic number fields. 1959.
- [Iwa73] Kenkichi Iwasawa. On Z_ℓ-extensions of algebraic number fields. Ann. of math, 98(246-326):7, 1973.
- [Joc94] Naomi Jochnowitz. A p-adic conjecture about derivatives of l-series attached to modular forms. Contemporary Mathematics, 165:239–239, 1994.
- [Maz72] Barry Mazur. Rational points of abelian varieties with values in towers of number fields. Inventiones mathematicae, 18(3-4):183–266, 1972.

i and T Nguyen Quang Do. Sur l'arithmétique des corps de nombres p- timinaire de Théorie des Nombres, Paris 1987–88, pages 155–200, 1990. ire and Marine Rougnant. A note on p-rational fields and the abc-conjecture. <i>f the American Mathematical Society</i> , 148(8):3263–3271, 2020. Sands. On the non-triviality of the basic Iwasawa λ -invariant for an infinitude quadratic fields. Acta Arithmetica, 65(3):243–248, 1993. or. On icosahedral Artin representations, ii. American journal of mathematics, 56, 2003.
ire and Marine Rougnant. A note on <i>p</i> -rational fields and the abc-conjecture. <i>f the American Mathematical Society</i> , 148(8):3263–3271, 2020. Sands. On the non-triviality of the basic Iwasawa λ -invariant for an infinitude quadratic fields. <i>Acta Arithmetica</i> , 65(3):243–248, 1993. or. On icosahedral Artin representations, ii. <i>American journal of mathematics</i> , 56, 2003.
Sands. On the non-triviality of the basic Iwasawa λ -invariant for an infinitude quadratic fields. Acta Arithmetica, 65(3):243–248, 1993. or. On icosahedral Artin representations, ii. American journal of mathematics, 66, 2003.
pr. On icosahedral Artin representations, ii. American journal of mathematics, 56, 2003.
. 2005.
A more general <i>abc</i> conjecture. <i>Internat. Math. Res. Notices</i> , (21):1103–1116,
Nashington. Introduction to cyclotomic fields, volume 83. Springer Science & ia, 1997.
A more general <i>abc</i> conjecture. <i>Internat. Math. Res. Notices</i> , (21):1103–11 Washington. <i>Introduction to cyclotomic fields</i> , volume 83. Springer Science ia, 1997.

(Karnataki) Chennai Mathematical Institute, H1, SIPCOT IT Park, Kelambakkam, Siruseri, Tamil Nadu 603103, India

Email address: adityak@cmi.ac.in

(Ray) Chennai Mathematical Institute, H1, SIPCOT IT Park, Kelambakkam, Siruseri, Tamil Nadu 603103, India

Email address: anwesh@cmi.ac.in