# Étale cohomology of Drinfeld half-spaces

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# Overview

# Fontaine-Messing period map

## Proposition (Proposition 6.1)

*Let $X$ be a semistable Stein formal scheme over $\mathscr{O}_K$. Let $r \geq 0$ and $\overline{X} := X_{\mathscr{O}_C}$. Recall that there is a natural Fontaine-Messing period map*

$$\alpha_{FM} : \mathrm{R}\Gamma_{\mathrm{syn}}(\overline{X}, \mathbf{Z}_p(r)) \otimes^L \mathbf{Q}_p \to \mathrm{R}\Gamma_{\text{ét}}(X_C, \mathbf{Z}_p(r)) \otimes^L \mathbf{Q}_p.$$

*This map is a strict quasi-isomorphism after truncation $\tau_{\leq r}$.*

Existence of this map follows by cohomological descent from the definition of integral Fontaine-Messing period map for a finite type formal scheme as defined by Fontaine and Messing. Required properties follow after noting that for some $N$, $p^N$ annihilates cohomology of the cone of $\alpha_{FM}$, and some careful analysis of the (locally convex) topology on source and target of $\alpha_{FM}$.

# Simplifying Syntomic Cohomology

Thus, we need to compute the crystalline geometric syntomic cohomology. A computation similar to one in previous talk shows the following strict quasi-isomorphism of distinguished triangles

$$\begin{array}{ccc}
\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\mathscr{O}_C}, \mathbf{Z}_p(r))_{\mathbf{Q}_p} & = & \mathrm{R}\Gamma_{\mathrm{syn}}(X_{\mathscr{O}_C}, \mathbf{Z}_p(r))_{\mathbf{Q}_p} \\
\downarrow & & \downarrow \\
\mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_C})_{\mathbf{Q}_p}^{\varphi=p^r} & \xrightarrow{\sim} & (\mathrm{R}\Gamma_{\mathrm{cr}}(X_k/\mathscr{O}_F^0)\widehat{\otimes}_{\mathscr{O}_F}\mathbf{A}_{\mathrm{st}})_{\mathbf{Q}_p}^{N=0,\varphi=p^r} \\
\downarrow & & \downarrow{\scriptstyle \gamma_{\mathrm{HK}}\otimes\iota} \\
\mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_C})_{\mathbf{Q}_p}/F^r & \xrightarrow{\sim} & (\mathrm{R}\Gamma_{\mathrm{dR}}(X)\widehat{\otimes}_{\mathscr{O}_K}\mathbf{A}_{\mathrm{cr},K})/F^r,
\end{array}$$

where $(-)_{\mathbf{Q}_p}$ denotes $(-) \otimes^L \mathbf{Q}_p$ and $\mathscr{O}_F^0$ denotes $\mathscr{O}_F$ with the log structure induced by $1 \to 0$. Both horizontal maps are defined in the proof of Proposition $3.48$ as essentially Künneth cup product maps. A discussion of this diagram is contained in Lemma $6.34$, equations $6.35$ and $6.36$.

# Differential Cohomology

In the case of Drinfeld half-space, the map $\gamma_{\mathrm{HK}}$ can be described better due to a simplification in its domain and codomain because of acyclicity of sheaves of differentials proved by Grosse-Klönne. This allows us to compute the syntomic, and hence, étale cohomology. The main focus of this talk is this acyclicity.

Let $\tilde{X} := \tilde{\mathbb{H}}^d_K, X := (\tilde{\mathbb{H}}^d_K)^\wedge$ denote the standard weak formal model, resp. formal model, of the Drinfeld half-space $\mathbb{H}^d_K$. Let $Y := X_0$ be the special fiber and $\overline{Y} := Y_{\overline{k}}$. Let $F^0$ be the set of irreducible components of $Y$. Let $T$ be the irreducible component with stabilizer $K^* GL_{d+1}(\mathscr{O}_K)$. It is the successive blowing up of $\mathbb{P}^d_k$ in all its $k$-rational linear subvarieties.
For $0 \leq j \leq d-1$, let $\mathscr{V}_0^j$ be the set of all $k$-rational linear subvarieties $Z$ of $\mathbb{P}^d_k$ with $\dim(Z) = j$ and let $\mathscr{V}_0 := \cup_{j=0}^{d-1} \mathscr{V}_0^j$. The set $T = T_{d-1} \to T_{d-2} \to \ldots \to T_0 = \mathbb{P}^d_k$ is defined inductively by defining $T_{j+1} \to T_j$ be the blowing up of $T_j$ in strict transforms in $T_j$ for all $Z \in \mathscr{V}_o^j$. The set $\mathscr{V}$ of all strict transforms in $T$ of elements of $\mathscr{V}_0$ is a set of NC divisors of $T$, and gives the log-structure. ($\mathscr{V}$ is in bijection with the set of vertices of the building associated to $PGL_{d+1}/k$.)

# Differential cohomology

Let $\tilde{\theta}_1, \ldots, \tilde{\theta}_d$ be the standard projective coordinate functions on $T_0 = \mathbb{P}_k^d$ and hence on $T$. For $i, j \in \{0, \ldots, d\}$ and $g \in G = GL_{d+1}(k)$, we call $g\mathrm{dlog}(\tilde{\theta}_i/\tilde{\theta}_j)$ a standard logarithmic differential 1-form on $T$. Exterior products of such forms will be called as standard logarithmic differential forms on $T$.

## Theorem (Grosse-Klönne, Proposition 6.3)

1. $H^i(T, \Omega^j) = 0, i > 0, j \geq 0.$
2. *The $k$-vector space $H^0(T, \Omega^j), j \geq 0$ is generated by standard logarithmic forms.*
3. $H^i_{\mathrm{cr}}(T/\mathscr{O}_F^0)$ *is torsion free, and*

$$H^i_{\mathrm{cr}}(T/\mathscr{O}_F^0) \otimes_{\mathscr{O}_F} k = H^i_{\mathrm{dR}}(T) = H^0(T, \Omega^j_T).$$

Since underlying scheme of $T$ is smooth, the crystalline cohomology is the same as log-crystalline cohomology with the log-structure given by $\mathscr{V}$.

# Line bundle cohomology

Define $U \subset GL_{d+1}$ to be the subgroup of unipotent upper triangular matrices. For $\tau \subset \{0, \ldots, d\}$, let $\tau^c$ be its complement set. For $\tau$ nonempty, let $V_{\tau,0}$ be the reduced closed subscheme of $T_0$ which is the common vanishing set of $(\tilde{\theta}_i)_{i \in \tau}$. Let $V_\tau \in \mathscr{V}$ be the strict transform of $V_{\tau,o}$ under $T \to T_0$. For any $\tau$, let $U_\tau = \{(a_{ij}) \in U(k) \mid a_{ij} = 0 \text{ if } i \neq j \text{ and } [j \in \tau^c \text{ or } \{i,j\} \subset \tau]\}$. Then, if we let $\mathcal{Y} := \{\tau \mid \emptyset \neq \tau \subsetneq \{0, \ldots, d\}\}$, and $\mathcal{N} := \{(\tau, u) \mid \tau \in \mathcal{Y}, u \in U_\tau\}$, we have a bijection $\mathcal{N} \cong \mathscr{V}$ given by $(\tau, u) \to u.V_\tau$.

For a divisor $D$ on a smooth connected $k$-scheme $S$, we denote by $\mathcal{L}_S(D)$ the associated line bundle on $S$. For an element $\bar{a} = (\bar{a}_1, \ldots, \bar{a}_d)$ of $\mathbb{Z}^d$, let $\bar{a}_0 = -\sum_{j=1}^{d} \bar{a}_j$, and for $\sigma \in \mathcal{Y}$, let $b_\sigma(\bar{a}) = -\sum_{j \in \sigma} \bar{a}_j$. Then for two more elements $\overrightarrow{n} = (n_1, \ldots, n_d)$ and $\overrightarrow{m} = (m_1, \ldots, m_d)$, we define the divisor on $T$

$$D(\bar{a}, \overrightarrow{n}, \overrightarrow{m}) := \sum_{\substack{\sigma \in \mathcal{Y} \\ 0 \notin \sigma}} (m_{|\sigma|} + b_\sigma(\bar{a})) \sum_{u \in U_\sigma} u.V_\sigma + \sum_{0 \in \sigma \in \mathcal{Y}} (n_{|\sigma|} + b_\sigma(\bar{a})) \sum_{u \in U_\sigma} u.V_\sigma .$$

# Relation with differential forms

There is a $U(k)$-stable filtration on the de Rham complex of logarithmic differential forms indexed by $\tau$. More precisely, for each $0 \leq s \leq d$, if $\mathcal{P}_s$ denotes the set of subsets of $\{1, \ldots, d\}$ of cardinality $s$, there is a $U(k)$-stable filtration $(F^\tau \Omega^s_T)_{\tau \in \mathcal{P}_s}$ on $\Omega^s_T$. On a large subset $T' \subset T$ whose $U(k)$-translates cover $T$, this is given explicitly by

$$F^\tau \Omega^s_{T'} = \bigoplus_{\substack{\tau' \in \mathcal{P}_s \\ \tau' \leq \tau}} \mathcal{O}_{T'} \bigwedge_{t \in \tau'} dz_t$$

where there is a lexicographical ordering on elements of $\mathcal{P}_s$.This explicit nature of filtration allows a computation of its graded pieces

$$Gr^\tau \Omega^s_T \cong \mathcal{L}_T(D(\bar{a}(\tau), \overrightarrow{0}, \overrightarrow{0}))$$

obtained by $f \wedge_{t \in \tau'} z_t \to f$ for $\bar{a}_i(\tau) = -1$ if $i \in \tau$ and $0$ otherwise.

# Relation with differential forms

In particular, this computation shows that vanishing of differential cohomology for $i > 0$ can be deduced from vanishing of the cohomology for the class of line bundles defined earlier. This argument is combinatorial, reducing to an induction argument after some technical steps. Base case corresponds to $\mathbb{P}^1$ where the combinatorics can be easily checked.
For $i = 0$, a similar but easier induction argument computes

$$\dim_k H^0(T, \mathcal{L}_T(D(\bar{a}(\tau), \overrightarrow{0}, \overrightarrow{0}))) = q^{\sum_{t\in\tau} i}$$

and then one can show linear indepdence of $\wedge_{t\in\tau} dlog(z_t)$ to see that they form a basis for $H^0(T, \Omega_T^s)$.
Finally, the base change property for $H^i_{\mathrm{cr}}$ is a consequence of the torsion freeness. To prove the torsion-free property, it suffices to check that the Newton polygon for $H^i_{\mathrm{cr}}(T/\mathscr{O}_F^0)$ coincides with the Hodge polygon for $H^i(T, \Omega_T^\bullet)$. These two have their endpoints the same. The latter has a single slope because of our computations above and Hodge spectral sequence, while the fact that the former has a single slope follows by a result of Grosse-Klónne.

# Acyclicity of sheaves of differentials

## Proposition (Grosse-Klönne, Proposition 6.5)

1. *For $j \geq 0$, we have topological isomorphisms*
$$H^i(X, \Omega_X^j) = 0 \text{ and } H^i(X, \Omega_X^j \otimes_{\mathscr{O}_K} k) = 0, i > 0,$$
$$H^0(X, \Omega_X^j) \otimes_{\mathscr{O}_K} k \cong H^0(X, \Omega_X^j \otimes_{\mathscr{O}_K} k).$$
2. $d = 0$ *on* $H^0(X, \Omega_X^j)$ *for* $j \geq 0$.

## Corollary (Corollary 6.6)

*There exist topological isomorphisms* $H^0(X, \Omega_X^j) \xrightarrow{\sim} H^i_{\mathrm{dR}}(X)$ *for all* $j \geq 0$.

## Proposition (Proposition 6.7)

*Let* $j \geq 0, s \in \mathbf{N}$*. Then for idealized log-schemes* $Y_s$ *defined earlier in section* 5.1*,* $H^i(Y_s, \Omega^j) = 0$ *for* $i > 0$ *and* $d = 0$ *on* $H^0(Y_s, \Omega^j)$*.*

# Relation to integral theory

For $V$ proper and log-smooth scheme over $k^0$ of Cartier type, we have the coboundaries and cocycles as subsheaves on $V_{ét}$, viz. $B_V^j := \mathrm{im}(d : \Omega_V^{j-1} \to \Omega_V^j)$ and $B_V^j := \ker(d : \Omega_V^j \to \Omega_V^{j+1})$. $V$ is called *ordinary* if for all $i, j \geq 0, H^i_{ét}(V, B^j) = 0$. Recall also the (logarithmic) Witt-de Rham complex $(W_n\Omega^\bullet_{\log,V})$ $W_n\Omega^\bullet_V$ of $V/k^0$. We write $\overline{V}$ for $V_{\overline{k}}$.

## Proposition (Lorenzon, Proposition 6.9)

*TFAE :*

- $V/k^0$ *is ordinary.*
- *For* $i, j \geq 0$*, the inclusion* $\Omega^j_{\overline{V},\log} \subset \Omega^j_{\overline{V}}$ *induces a canonical isomorphism of* $\overline{k}$*-vector spaces*

$$H^i_{ét}(\overline{V}, \Omega^j_{\log}) \otimes_{\mathbf{F}_p} \overline{k} \xrightarrow{\sim} H^i_{ét}(\overline{V}, \Omega^j).$$

# Relation to integral theory

## Proposition (Continued)

- *For* $i, j, n \geq 0$*, the canonical maps*

$$H^i_{\acute{e}t}(\overline{V}, W_n\Omega^j_{\log}) \otimes_{\mathbf{Z}/p^n} W_n(\bar{k}) \to H^i_{\acute{e}t}(\overline{V}, W_n\Omega^j),$$

$$H^i_{\acute{e}t}(\overline{V}, W\Omega^j_{\log}) \otimes_{\mathbf{Z}_p} W(\bar{k}) \to H^i_{\acute{e}t}(\overline{V}, W\Omega^j),$$

  *are isomorphisms.*

- *For* $i, j \geq 0$*, the Frobenius*

$$\mathrm{F} : H^i_{\acute{e}t}(\overline{V}, W\Omega^j) \to H^i_{\acute{e}t}(\overline{V}, W\Omega^j)$$

  *is an isomorphism.*

In fact, the argument with Newton and Hodge polygons shows that $T$ is ordinary. $\mathbb{H}^d_K$ is a pro-ordinary log scheme.

# $\mathbb{H}_K$ as a pro-ordinary log scheme

We now drop the assumption on $V$ to be proper.

**Lemma (Lemma 6.13)**

*Assume that* $H^i_{\acute{e}t}(V, \Omega^j) = 0$ *and that* $d = 0$ *on* $H^0_{\acute{e}t}(V, \Omega^j)$ *for all* $i \geq 1$ *and* $j \geq 0$*. Then* $V$ *is ordinary (in the sense of vanishing of cohomology of coboundary sheaves).*

Equipped with these results, the next key step is the following proposition.

**Proposition (Proposition 6.23)**

*Let* $j \geq 0$*. Let* $S$ *be a topological* $\mathscr{O}_K$*-module and let* $R$ *be a topological* $W(k)$*- or* $W(\overline{k})$*-module. Assume that* $S$ *and* $R$ *are topologically orthonormalizable.*

# Cohomology of differentials again

1. *The following natural maps are strict quasi-isomorphisms* (in $\mathscr{D}(\mathrm{Ind}(PD_?))$, with $? = K, F, \mathbf{Q}_p$)

$$H^0(X, \Omega^j_{X,n})\widehat{\otimes}_{\mathscr{O}_{K,n}} S_n \xrightarrow{\sim} \mathrm{R}\Gamma(X, \Omega^j_{X,n})\widehat{\otimes}_{\mathscr{O}_{K,n}} S_n,$$
$$H^0_{\text{ét}}(Y, W_n\Omega^j)\widehat{\otimes}_{\mathscr{O}_{F,n}} R_n \xrightarrow{\sim} \mathrm{R}\Gamma_{\text{ét}}(Y, W_n\Omega^j)\widehat{\otimes}_{\mathscr{O}_{F,n}} R_n,$$
$$\mathrm{R}\Gamma_{\text{ét}}(\overline{Y}, W_n\Omega^j_{\log})\widehat{\otimes}_{\mathbf{Z}/p^n} R_n \xrightarrow{\sim} \mathrm{R}\Gamma_{\text{ét}}(\overline{Y}, W_n\Omega^j)\widehat{\otimes}_{W_n(\overline{k})} R_n,$$
$$H^0_{\text{ét}}(\overline{Y}, W_n\Omega^j_{\log})\widehat{\otimes}_{\mathbf{Z}/p^n} R_n \xrightarrow{\sim} \mathrm{R}\Gamma_{\text{ét}}(\overline{Y}, W_n\Omega^j_{\log})\widehat{\otimes}_{\mathbf{Z}/p^n} R_n.$$

2. $d = 0$ *on* $H^0_{\text{ét}}(X, \Omega^j_{X,n})\widehat{\otimes}_{\mathscr{O}_{K,n}} S_n$ *and on* $H^0_{\text{ét}}(Y, W_n\Omega^j_Y)\widehat{\otimes}_{\mathbf{Z}/p^n} R_n$.
3. *The following natural map is a strict quasi-isomorphism*

$$\bigoplus_{j\geq r} H^0(X, \Omega^j_{X,n})[-j] \xrightarrow{\sim} F^r\mathrm{R}\Gamma_{\mathrm{dR}}(X_n), \quad r \geq 0.$$

In particular, a consequence (Corollary $6.25$) is that $H^i(X, \Omega^j_X) = 0$ for $i > 0$ and $H^0(X, \Omega^j_X) \cong H^j_{\mathrm{dR}}(X)$. (Mittag-Leffler property is important in deriving this.)

# de Rham cohomology of the model and generic fiber

Another key step is to understand the de Rham cohomology of the model. Define the map

$$\iota_Y : H^i_{\acute{e}t}(Y, W\Omega^\bullet_Y) \cong H^i_{\mathrm{cr}}(Y/\mathscr{O}^0_F, F) \xleftarrow{\sim} H^i_{\mathrm{rig}}(Y/\mathscr{O}^0_F) \xrightarrow{\iota_{HK}} H^i_{\mathrm{rig}}(Y/\mathscr{O}^\times_K).$$

## Proposition (Proposition 6.27)

1. *The above map induces an injection*

$$\iota_Y : H^i_{\acute{e}t}(Y, W\Omega^\bullet_Y) \otimes_{\mathscr{O}_F} K \hookrightarrow H^i_{\mathrm{rig}}(Y/\mathscr{O}^\times_K).$$

2. *The canonical map*

$$H^i_{\mathrm{dR}}(X) \otimes_{\mathscr{O}_K} K \to H^i_{dR}(X_K)$$

*is injective.*

These key results are used in the proof of following result which relates the Witt-de Rham cohomology with generalised Steinberg representations.

# Relation of Witt-de Rham cohomology with Steinberg representations

## Proposition (Proposition 6.28)

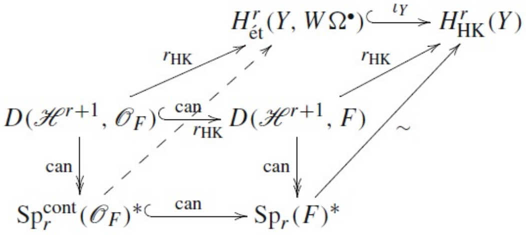
*Let $r \geq 0$.*

1. *We have natural isomorphisms of locally convex topological $\mathbf{Q}_p$-vector spaces (more precisely, weak duals of Banach spaces)*
   (a) $H^0(Y, W\Omega^r)\otimes_{\mathcal{O}_F} F \simeq H^r(Y, W\Omega^\bullet)\otimes_{\mathcal{O}_F} F \simeq \mathrm{Sp}_r^{\mathrm{cont}}(F)^*$,
   (b) $H^0_{\text{ét}}(Y, W\Omega^r_{\log})\otimes\mathbf{Q}_p \simeq \mathrm{Sp}_r^{\mathrm{cont}}(\mathbf{Q}_p)^*$,
   (c) $H^0(X, \Omega^r)\otimes_{\mathcal{O}_K} K \simeq H^r_{\mathrm{dR}}(X)\otimes_{\mathcal{O}_K} K \simeq \mathrm{Sp}_r^{\mathrm{cont}}(K)^*$,
   (d) $H^0_{\text{ét}}(\overline{Y}, W\Omega^r_{\log})\otimes\mathbf{Q}_p \simeq \mathrm{Sp}_r^{\mathrm{cont}}(\mathbf{Q}_p)^*$.
   *They are compatible with the canonical maps between Steinberg representations and with the isomorphisms*

$$H^r_{\mathrm{dR}}(X_K) \simeq \mathrm{Sp}_r(K)^*, \quad H^r_{\mathrm{HK}}(X) \simeq \mathrm{Sp}_r(F)^*$$

There is an integral version as well.

# Relation of Witt-de Rham cohomology with Steinberg representations




The (first) continuous regulator $r_{\text{HK}}$ is defined by integrating crystalline Hyodo-Kato Chern classes $c_1^{\text{HK}}$ defined in the next section (Iovita-Spiess). In particular, commuting of various arrows gives the existence of broken arrow. Since it makes the square formed by it commute, it is an injection. Proposition $6.27$ and Theorem $5.11$ together with an examination of $G$-bounded vectors of $Sp_r(K)^*$ shows that it is an isomorphism after inverting $p$. The fact that it is an integral isomorphism can be established with some more care.

# Computation of Syntomic cohomology

## Lemma (Lemma 6.37)

*The cohomology of* $[(R\Gamma_{\mathrm{cr}}(Y/\mathscr{O}_F^0)\hat{\otimes}_{\mathscr{O}_F}\hat{\mathbf{A}}_{\mathrm{st}})_{\mathbf{Q}_p}]^{N=0,\varphi=p^r}$ *is classical, and there are natural topological isomorphisms*

$$H^r([(R\Gamma_{\mathrm{cr}}(Y/\mathscr{O}_F^0)\hat{\otimes}_{\mathscr{O}_F}\hat{\mathbf{A}}_{\mathrm{st}})_{\mathbf{Q}_p}]^{N=0,\varphi=p^r}) \cong H^0_{\acute{e}t}(\overline{Y}, W\Omega^r_{\log}) \otimes \mathbf{Q}_p,$$

$$H^{r-1}([(R\Gamma_{\mathrm{cr}}(Y/\mathscr{O}_F^0)\hat{\otimes}_{\mathscr{O}_F}\hat{\mathbf{A}}_{\mathrm{st}})_{\mathbf{Q}_p}]^{N=0,\varphi=p^r}) \cong H^0_{\acute{e}t}(\overline{Y}, W\Omega^{r-1}_{\log}\hat{\otimes}_{\mathbf{Z}_p}\mathbf{A}_{\mathrm{cr}}^{\varphi=p}) \otimes \mathbf{Q}_p.$$

This lemma achieves the program of simplification of the earlier diagram on slide $4$, and gives the following as corollary after some more work.

## Corollary (Corollary 6.42)

*The cohomology of* $\mathrm{R}\Gamma_{\mathrm{syn}}(\overline{X}, \mathbf{Z}_p(r))_{\mathbf{Q}_p}$ *is classical and there exists a topological isomorphism*

$$H^r(\mathrm{R}\Gamma_{\mathrm{syn}}(\overline{X}, \mathbf{Z}_p(r))_{\mathbf{Q}_p}) \cong H^0_{\acute{e}t}(\overline{Y}, W\Omega^r_{\log}) \otimes \mathbf{Q}_p.$$

# Main Theorem

## Theorem (CDN, Theorem 6.45)

*Let $r \geq 0$.*

1. *There is a natural topological isomorphism of $G \times \mathscr{G}_K$-modules*

$$H^r_{\text{ét}}(X_C, \mathbf{Q}_p(r)) \simeq \mathrm{Sp}^{\text{cont}}_r(\mathbf{Q}_p)^* \simeq \mathrm{Sp}_r(\mathbf{Z}_p)^* \hat{\otimes} \mathbf{Q}_p.$$

2. *There are natural topological isomorphisms of $G$-modules*

$$H^r_{\text{dR}}(X)_K \simeq \mathrm{Sp}^{\text{cont}}_r(K)^*, \quad H^r_{\text{cr}}(Y/\mathscr{O}^0_F)_{\mathbf{Q}_p} \simeq \mathrm{Sp}^{\text{cont}}_r(F)^*,$$

$$H^i_{\text{ét}}(\overline{Y}, W\Omega^r_{\overline{Y},\log})_{\mathbf{Q}_p} \simeq \begin{cases} \mathrm{Sp}^{\text{cont}}_r(\mathbf{Q}_p)^* & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

The bulk of work for both of these is already done. Point $2$ follows from Theorem $6.28$ and Proposition $6.23$. Point $1$ follows from computation of logarithmic Witt-de Rham cohomology in theorem $6.28$ as well, after comparison results lemma $6.32$ and corollary $6.44$.

# Main Theorem continued

3. *The regulator maps*

$$r_{\mathrm{dR}} : M(\mathscr{H}^{d+1}, K) \to H^r_{\mathrm{dR}}(X)_K, \quad r_{\mathrm{HK}} : M(\mathscr{H}^{d+1}, F) \to H^r_{\mathrm{cr}}(Y/\mathscr{O}_F^0)_{\mathbf{Q}_p},$$
$$r_{\log} : M(\mathscr{H}^{d+1}, \mathbf{Q}_p) \to H^0_{\mathrm{\acute{e}t}}(\overline{Y}, W\Omega^r_{\log})_{\mathbf{Q}_p}$$

*are strict surjective maps (of weak duals of Banach spaces), $G \times \mathscr{G}_K$-equivariant, compatible with the isomorphisms in (1) and (2), and their kernels are equal to the space of degenerate measures (defined as the intersection of the space of measures with the set of degenerate distributions).*

4. *The natural map*

$$H^r_{\mathrm{\acute{e}t}}(X_C, \mathbf{Q}_p(r)) \to H^r_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p(r))$$

*is an injection and identifies $H^r_{\mathrm{\acute{e}t}}(X_C, \mathbf{Q}_p(r))$ with the $G$-bounded vectors of $H^r_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p(r))$.*

Point $4$ follows from an analysis of $G$-bounded vectors and Proposition $6.27$. Point $3$ follows from the construction and properties of symbol maps.

# Differential symbols

We know by Schneider-Stuhler's results that there is a natural isomorphism $H^r_{\mathrm{dR}}(X_K) \cong Sp_r(K)^*$ of $G$-representations. Iovita-Spiess prove that $H^r_{\mathrm{dR}}(X_K)$ is generated by standard symbols, i.e. cup products of symbols of $K$-rational hyperplanes. $Sp_r(K)^*$ is generated by standard symbols, too, considered as invertible functions on $X_K$, and this isomorphism is compatible with symbols.

In particular, the isomorphism $H^r_{\mathrm{HK}}(X_k) \cong Sp_r(F)^*$, used for the pro-étale computations, factors through this isomorphism after defining the regulator map $r_{HK}$ using the overconvergent Hyodo-Kato Chern classes $c_1^{\mathrm{HK}}$. We also saw their appearance of these during the étale cohomology computations of Theorem $6.28$. We state their definitions and compatibility properties briefly.

# Differential symbols

Let $X$ be a semistable formal scheme over $\mathscr{O}_K$. Let $M$ be the sheaf of monoids on $X$ defining the log-structure. Then, there exist crystalline first Chern class maps

$$c_1^{\mathrm{st}} : H^0(X_K, \mathscr{O}_{X_K}^*) \to \mathrm{R}\Gamma_{\mathrm{cr}}(X, \mathscr{J}^{[1]})[1],$$

$$c_1^{\mathrm{HK}} : H^0(X_K, \mathscr{O}_{X_K}^*) \to \mathrm{R}\Gamma_{\mathrm{cr}}(X_0, \mathscr{J}^{[1]})[1]$$

induced from crystalline first Chern class maps of complexes of certain ideal sheaves on $X_{\acute{e}t}$ that were defined by Tsuji. ("p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case", section $2.2.3$.) The Hyodo-Kato chern classes can also be defined using the Witt-de Rham complex, i.e. there exist Chern class maps

$$c_1^{\log} : H^0(X_K, \mathscr{O}_{X_K}^*) \to \mathrm{R}\Gamma_{\text{ét}}(X_0, W\Omega^\bullet_{X_0,\log})[1],$$

$$c_1^{\mathrm{HK}} : H^0(X_K, \mathscr{O}_{X_K}^*) \to \mathrm{R}\Gamma_{\text{ét}}(X_0, W\Omega^\bullet_{X_0})[1].$$

# Differential symbols

Similarly, there exists a de Rham first Chern class map

$$c_1^{\mathrm{dR}} : H^0(X_K, \mathscr{O}_{X_K}^*) \to \mathrm{R}\Gamma_{\mathrm{dR}}(X)[1].$$

By the canonical map $\mathrm{R}\Gamma_{\mathrm{dR}}(X) \to \mathrm{R}\Gamma_{\mathrm{dR}}(X_K)$, this map is compatible with rigid analytic first Chern class map. Finally, we can define overconvergent analogues of these crystalline Hyodo-Kato classes and the rigid de Rham classes for $X$ semistable Stein weak formal scheme over $\mathscr{O}_K$ because of Stein property (crystalline to overconvergent change of topology is well-behaved, and so are overconvergent differential forms).

## Lemma (Lemma 7.8)

*For $X$ semistable Stein weak formal scheme over $\mathscr{O}_K$, the Hyodo-Kato maps*

$$\iota_{\mathrm{HK}} : H^1_{\mathrm{cr}}(X_0/\mathscr{O}_F^0, F) \to H^1_{\mathrm{dR}}(\hat{X}_K), \iota_{\mathrm{HK}} : H^1_{\mathrm{rig}}(X_0/\mathscr{O}_F^0) \to H^1_{\mathrm{dR}}(X_K)$$

*are compatible with the Chern class maps.*

Thank you!